

Playsheet 9

Infinity is Weird! (Take Two)

MATH 130-02
Thursday, February 26, 2009

Directions: Groups should consist of three or four people. Work together on each problem; do not delegate different problems to different people. Submit one **neatly written** write-up per group. Remember to use complete sentences as appropriate and explain your reasoning. That is, **show your work!**

1. Recall the game of **dodgeball** from the the first worksheet. Play it once or twice to remind yourself of the strategy.

1						
2						
3						
4						
5						
6						

Player 1

1						
2						
3						
4						
5						
6						

Player 1

1	2	3	4	5	6

Player 2

1	2	3	4	5	6

Player 2

2. Now consider the same game, but with digits 0-9 instead of X's and O's. Also, imagine that the game board goes on forever instead of stopping after six squares. You won't be able to see the whole board, but you can imagine it! Now play the game with these conditions. Who wins?

1									
2									
3									
4									
5									
6									
7									
8									
9									
10									

Player 1

1	2	3	4	5	6	7	8	9	10

Player 2

3. **How long are the irrationals?** In this series of exercises, we will try to figure out what percentage of the numbers between 0 and 1 are irrational. Trust me.

(a) Consider the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. (This is known as a **geometric** sum.)

- i. Compute the three sums $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, and $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$. Simplify your answers, but leave them as fractions.

- ii. Without computing it, make a guess as to what $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$ is (in fraction form).

(OVER)

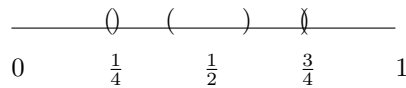
iii. Now generalize: what is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n}$ (in fraction form)?

iv. What happens to this as n gets larger and larger? That is, what is the value of the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$?

An **interval** is the set of all real numbers between two given numbers. For example, $(3,5)$ represents the interval of real numbers between 3 and 5.

- (b) Now consider the rational numbers between 0 and 1. We have shown that there is a one-to-one correspondence between these numbers and the natural numbers $1, 2, 3, \dots$, so we can talk about the “first” rational number in our list, the “second” rational number in our list, and so on. Imagine now a little interval of width 0.01 around the first number in your list. (Imagine this on a numberline). For example, if your first rational number is 0.5, then the interval would go from 0.495 to 0.505. Now put an interval of width 0.001 around the second number in your list, then an interval of width 0.0001 around the third number, and so on.

In the figure below, I have begun my list of fractions with $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}$. Note that the intervals around these points are shrinking as I move down the list.



When you are finished, every rational number is inside one of these intervals. What is the total combined width of **all** of the intervals? (Note that there are infinitely many!)

- (c) Suppose that, instead of starting with 0.01, you started with 0.001. Then what would the combined width be?

- (d) Once more: what would be the combined width if you started with 0.0001?

- (e) Now summarize: What is the trend here? That is, how small could we make the total width?

- (f) Given your conclusions above, what percentage of the interval $(0, 1)$ is made up of rational numbers? Irrational numbers?