

Chapter 7

Functions

This unit defines and investigates functions as algebraic objects. First, we define functions and discuss various means of representing them. Then we introduce operations on functions and examine which field properties hold for the operations, with emphasis on composition of functions.

Objectives

- To define function and introduce operations on the set of functions
- To investigate which of the field properties hold in the set of functions
- To explore various ways of representing functions
- To examine graphs of “basic” functions and simple transformations of them
- To study inverse functions in the context of the field properties and introduce algebraic methods for determining whether a function has an inverse and, if so, to find it
- To introduce special functions (polynomial, exponential, logarithmic, and sequences)

Terms

- function
 - geometric sequence
- domain
 - zero (of a function)
- range
 - intercept
- sequence
 - vertical line test
 - term
 - equality of functions
 - recursively defined sequence
 - operations on functions
 - arithmetic sequence
 - sum (difference)

- product (quotient)
- composition
- one-to-one functions
- horizontal line test
- inverse functions

7.1 Functions

Most students are already familiar with the idea of a function. Here we formalize this idea and examine which field properties hold for a collection of functions.

Definition 7.1.1. A **function** f from a set X to a set Y is a pairing of elements from X with elements from Y in such a way that every element of X is associated with exactly one element of Y ; we write $f : X \rightarrow Y$.

If $y \in Y$ is paired with $x \in X$ in the function f , we write $f(x) = y$ and say that y is the **image** of x under f . We sometimes refer to x as the **input** to f and y as the **output** from f .

Cautionary note: The symbol $f(x)$ does NOT signify multiplication, even though the notation is similar. You will need to be alert for context to decide which is intended. Generally, we will reserve the letters f, g , and h for functions.

The set X is called the **domain** of f , and the set of all images

$$R = \{y \mid y = f(x) \text{ for some } x \in X\}$$

is called the **range** of f . The domain is usually considered as the largest subset of the real numbers for which the function is defined, unless otherwise stated. The range is usually somewhat hard to find, but a graph of the function can help, as we will see in another section.

Most of the functions we are concerned with involve performing operations on real numbers. To determine the domain, we must determine which operations will *yield* real numbers. Are we trying to find a multiplicative inverse for 0? Are we trying to find a square root of a negative number? Or are we just adding and multiplying real numbers?

Example 7.1.2. Determine the domain and range of each function, and compute $f(1)$ and $f(-4)$, if possible.

1. $f(x) = \frac{1}{x}$.

Remember that $\frac{1}{x}$ means the multiplicative inverse of x . This means that given the input x , the function f returns its multiplicative inverse as the output. This can be done as long as $x \neq 0$; therefore, the domain of f is $\{x \in \mathbb{R} \mid x \neq 0\}$.

For the range, we need to find out what values of $f(x)$ can be obtained. Set $y = f(x)$ for convenience. Then we need to know what values of y can be obtained from $y = \frac{1}{x}$.

If $y = \frac{1}{x}$, then also $x = \frac{1}{y}$, so it is clear that $y \neq 0$ for the same reason that $x \neq 0$.

However, any other value of y is obtainable since if $y \neq 0$, then we can set $x = \frac{1}{y}$, and

then $f(x) = \frac{1}{1/y} = y$. Therefore, the range of f is $\{y \in \mathbb{R} \mid y \neq 0\}$.

$$f(1) = \frac{1}{1} = 1, \text{ and } f(-4) = \frac{1}{-4} = -\frac{1}{4}.$$

2. $f(x) = \sqrt{x}$.

Recall that \sqrt{x} means a positive number a such that $a^2 = x$. However, we know from earlier work that if $a \in \mathbb{R}$, then $a^2 \geq 0$. Therefore, we must have $x \geq 0$. It is a fact (that we have not discussed) that every positive real number has a positive square root. This means that the domain of f is $\{x \in \mathbb{R} | x \geq 0\} = [0, \infty)$.

If y is in the range of f , then $f(x) = \sqrt{x} = y$ for some real number x , so $y \geq 0$. (Remember, $\sqrt{x} \geq 0$ for all $x \geq 0$.) Conversely, if $y \geq 0$, then let $x = y^2$. We get $f(x) = \sqrt{x} = \sqrt{y^2} = |y| = y$ since $y \geq 0$.

That is, y is in the range of f if and only if $y \geq 0$, so the range of f is $\{y \in \mathbb{R} | y \geq 0\} = [0, \infty)$.

$f(1) = \sqrt{1} = 1$, but $f(-4)$ is not defined in the real numbers.

3. $f(x) = \frac{x+2}{x^2-4}$

We have $f(x) = \frac{x+2}{(x+2)(x-2)}$. Remember that $\frac{a}{a} = 1$, provided that $a \neq 0$; when we are finding the domain, we may not simplify. We can also represent f by

$$f(x) = (x+2) \cdot \frac{1}{x+2} \cdot \frac{1}{x-2}.$$

We are now in a position to determine the domain of f . This function requires finding multiplicative inverses for $x+2$ and $x-2$; if $x = -2$, then $x+2$ does not have a multiplicative inverse (so f is not defined), and if $x = 2$ then $x-2$ does not have a multiplicative inverse (so f is not defined). Since $x = 2$ and $x = -2$ are the only values of x that will not yield a real number for y , they are the only real numbers not in the domain of f . Therefore, the domain of f is

$\{x \in \mathbb{R} | x \neq 2 \text{ and } x \neq -2\}$.

$$f(1) = \frac{1+2}{1^2-4} = -1. \quad f(-4) = \frac{-4+2}{(-4)^2-4} = -\frac{1}{6}.$$

We are not equipped to determine the range right now.

4. $f(x) = 2x^3 - 5$.

When we are given x , we must first find $x^3 = x \cdot x \cdot x$, then multiply it by 2, and then subtract 5 (recalling the order of operations). Since all of these operations may be performed on any real number, the domain of f is \mathbb{R} . The range is also \mathbb{R} since all real numbers have a real cube root: given y , let $x = \sqrt[3]{(y+5)/2}$. Then $f(x) = 2(\sqrt[3]{(y+5)/2})^3 - 5 = 2[(y+5)/2] - 5 = y + 5 - 5 = y$.

$$f(1) = 2(1)^3 - 5 = -3. \quad f(-4) = 2(-4)^3 - 5 = -133.$$

The examples above illustrate two ways that a number x can fail to be in the domain of a function: x can cause a division by 0 (which is not defined), or x can cause us to try to find a (real) square root for a negative number, which is also not defined. There are other

ways that x can fail to be in the domain of a function, as well, and we will discuss some of these later.

We have a variety ways to represent a function:

1. Tabular representation: We can arrange the information in a table, as follows:

x	$f(x)$
-2	-10
-1	-7
0	-4
1	-1
2	2

This works fine for simpler functions or for data gathered experimentally, but it can be awkward to extrapolate from.

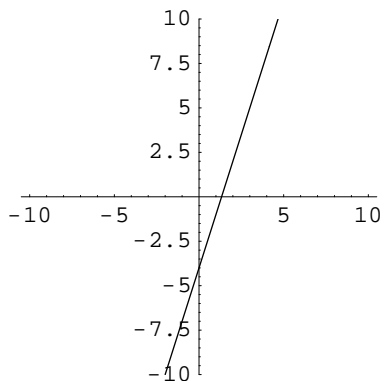
2. Ordered pair representation: We can think of functions as ordered pairs in $X \times Y$, where every element of X appears in exactly one ordered pair:

$$f = \{(x, y) | f(x) = y \text{ and } x \in X\}.$$

For example, the function above is $f = \{(x, 3x - 4) | x \in \mathbb{R}\}$ if we extrapolate, or $f = \{(-2, -10), (-1, -7), (0, -4), (1, -1), (2, 2)\}$ if we do not. Notice that $(x, y) \in f$ if and only if $y = f(x)$.

This is perhaps the most natural representation from the point of view of the definition; functions are *defined* as pairings, and here we are just being explicit about what the pairings are.

3. Graphical representation: Because we can think of functions as ordered pairs, we can graph them, too! We plot points on a Cartesian coordinate system until we have enough to figure out what the graph looks like. (The points we plot are just those corresponding to ordered pairs that belong to the function.) We can be fooled, though, so we must be careful. We can also determine whether some relations are functions by using the **vertical line test** (see below), but we have to be able to graph the relation to begin with. Your calculator can help you with some. We will look at graphing in more detail later. The graph of the function in parts 1 and 2 is below, with a few specific points plotted, and the rest filled in by extrapolation.



4. Algebraic representation: The examples above can be interpreted in terms of some formula or equation, namely, $f(x) = 3x - 4$.
5. Verbal representation: A function can also be given by a description, such as “Each person is paired with the table at which that person is sitting.”

Example 7.1.3. Consider $f : \{1, 2, 3, 4\} \rightarrow \{0, 1\}$ given by $f(1) = 1, f(2) = 0, f(3) = 1,$ and $f(4) = 0$. Then f is a function with domain $\{1, 2, 3, 4\}$ and range $\{0, 1\}$. The function f can also be represented as a set of ordered pairs $f = \{(1, 1), (2, 0), (3, 1), (4, 0)\}$; note that each element of the domain appears in exactly one ordered pair.

Example 7.1.4. Let $g : \{1, 2, 3, 4\} \rightarrow \{0, 1\}$ satisfy $g(1) = 0, g(2) = 1, g(3) = 1, g(4) = 0,$ and $g(4) = 1$. Then g is not a function since 1 is paired with both 0 and 1. As a set of ordered pairs, $g = \{(1, 0), (2, 1), (1, 1), (3, 0), (4, 1)\}$. Since the element 1 is used twice as a first-coordinate, g is not a function.

Example 7.1.5. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be given by $h(x) = x^2 - 3x + 1$. Then h is a function with domain \mathbb{R} and a range that we are not equipped to determine at present. We have

$$h(3) = 3^2 - 3(3) + 1 = 1 \text{ and } h(-2) = (-2)^2 - 3(-2) + 1 = 11.$$

Example 7.1.6. Function evaluation is to be taken quite literally. A formula for $f(x)$ means that *no matter what x is*, its value is to be substituted unchanged into the formula. Consider $f(x) = \frac{x}{x+1}$. $f(w-3) = \frac{w-3}{(w-3)+1}$.

In the following sections we will consider some special functions that are noteworthy; in particular, we'll explore polynomial functions, exponential functions, logarithmic functions, and sequences.

Exercises

Determine the domain of each function and express your answer in interval notation.

$$1. f(x) = \frac{1}{2x+5}$$

$$4. f(x) = \frac{x^2}{\sqrt{x-3}}$$

$$2. f(x) = \frac{x-1}{x+1}$$

$$5. f(x) = \sqrt{x^2-9}$$

$$3. f(x) = \frac{x^2-5x-3}{x^2-5x-6}$$

$$6. f(x) = x^2 - 5x$$

Evaluate each function as indicated and simplify.

$$7. \text{ For } f(x) = \sqrt{x^2 - 2x + 2}, \text{ find } f(0), f(-2), \text{ and } f(3).$$

8. For $p(t) = \frac{t}{t+1}$, find $p(4)$, $p(-5.2)$, $p(100)$, and $p(0)$.
9. For $x(t) = t^t$, find $x(1)$, $x(-3)$, and $x(2)$.
10. For $f(x) = x^2$, find $f(2)$, and $\frac{f(x+h) - f(x)}{h}$.
11. Given the function $f(x) = \frac{1}{1+x^2}$, create a tabular representation with 6 values for x (of your choosing). Then plot the 6 points from your table and connect them with a smooth curve.
12. In professional football games, the analysts often discuss the “hang time” of a punt. If the hang time is T seconds, then it can be shown that a reasonable estimate for the height of the football after t seconds have elapsed is $h(t) = 16t(T - t)$ feet. Also, since the time the ball spends as much time going up as coming down (roughly), the maximum height occurs at $t = \frac{1}{2}T$. If a punt has a hang time of 5 seconds, how high does it go?
13. The same equation as in the previous problem can be used to model jumps of basketball players. If a basketball player has a hang time of 1 second, how high did the player jump? (Compare to Clyde Drexler’s 44 inch vertical leap.)
14. Let $f(x) = 3x^3 - 5x^2 + 7$, and let $g(x) = 3x^2 - 2x + 5$.
 - (a) Compute the new function $h(x) = f(x) + g(x)$ and simplify.
 - (b) Compute $h(2)$.
 - (c) Compute $f(2)$ and $g(2)$ and add them.
 - (d) What do you notice about your answers for parts (b) and (c)?

7.2 Sequences

We consider special type of function known as a sequence; sequences are functions with a special (simple) domain.

Definition 7.2.1. A **sequence** is a function whose domain is the set of positive integers.

We usually represent a sequence by listing its values in order: $a(1), a(2), \dots$. The numbers are the **terms** of the sequence.

Example 7.2.2. The formula $a(n) = 1/n$, with $n \in \mathbb{N}$, gives the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ since $a(1) = 1, a(2) = \frac{1}{2}$, etc.

Example 7.2.3. The formula $b(n) = 3n + 1$ gives the sequence $4, 7, 10, 13, 16, \dots$

Instead of $a(n)$, we often use subscripts and write a_n to represent a general term (n th term) and list $a_1, a_2, \dots, a_n, \dots$ to represent a listing of the terms of the sequence in order. In the examples above, we had the general terms $a_n = 1/n$ and $b_n = 3n + 1$. (The letter we choose corresponds to the name of the function.)

Definition 7.2.4. The notation $\{a_n\}$ stands for the sequence whose n th term is a_n .

Example 7.2.5. List the first five terms of the sequence $\{a_n\} = \{(n-1)/n\}$.

Solution: We have $a_1 = \frac{1-1}{1} = 0$, $a_2 = \frac{2-1}{2} = \frac{1}{2}$, $a_3 = \frac{3-1}{3} = \frac{2}{3}$, $a_4 = \frac{3}{4}$, and $a_5 = \frac{4}{5}$.

Example 7.2.6. List the first seven terms of $\{c_n\}$, where $c_n = n$ if n is even, and $c_n = 1/n$ if n is odd.

Solution: This function is defined piecewise: $c_n = \begin{cases} n & n \text{ even} \\ 1/n & n \text{ odd} \end{cases}$. Thus we have $c_1 = \frac{1}{1} = 1$, $c_2 = 2$, $c_3 = \frac{1}{3}$, $c_4 = 4$, $c_5 = \frac{1}{5}$, $c_6 = 6$, and $c_7 = \frac{1}{7}$.

Definition 7.2.7. For a positive integer n , we define $n!$ by $n! = n(n-1)(n-2) \cdots (3)(2)(1)$. In other words, $n!$ is the product of the first n natural numbers. We also adopt the convention that $0! = 1$. The symbol $n!$ is pronounced “ n factorial” or “factorial n .”

Notice that $(n+1)! = (n+1)n!$; this observation will prove useful later on.

Example 7.2.8. Simple calculations yield the following.

- $5! = (5)(4)(3)(2)(1) = 120$
- $13! = (13)(12)(11) \cdots (3)(2)(1) = 6,227,020,800$

Definition 7.2.9. A **recursively defined sequence** is a sequence whose terms are specified by a formula involving the preceding terms.

Example 7.2.10. The Fibonacci sequence is given by

$$a_1 = 1, a_2 = 1, \dots, a_{n+2} = a_{n+1} + a_n.$$

List the first eight terms.

Solution: We have $a_1 = 1$ and $a_2 = 1$ given. We compute $a_3 = a_2 + a_1 = 1 + 1 = 2$, $a_4 = a_3 + a_2 = 2 + 1 = 3$, $a_5 = a_4 + a_3 = 3 + 2 = 5$, $a_6 = a_5 + a_4 = 5 + 3 = 8$, $a_7 = 8 + 5 = 13$, and $a_8 = 13 + 8 = 21$. Thus the Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, \dots$

Example 7.2.11. Find the first five terms of the sequence $f_1 = 1, f_{n+1} = (n + 1)f_n$.

Solution: We have that $f_2 = 2(1) = 2$, $f_3 = 3(2) = 6$, $f_4 = 4(6) = 24$, and $f_5 = 5(24) = 120$. Looking at this pattern, we see that we have $f_n = n!$, the factorial sequence.

Frequently a sequence is indicated by a partial listing of its terms and we desire a formula for the general term. This problem is technically unsolvable *without more information*. In fact, for any finite listing of sequence terms, there are infinitely many (correct) completing patterns and general terms.

Example 7.2.12. Find the next six terms and explain the pattern based on the partial sequence $1, 2, 4, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}$.

Solution 1: Using the pattern of doubling, we have $1, 2, 4, 8, 16, 32, 64, 128, 256 \dots, 2^{n-1}, \dots$. In this case, a closed formula is $a_n = 2^{n-1}$.

Solution 2: Using the pattern of adding one more to each successive difference, we have $1, 2, 4, 7, 11, 16, 22, 29, 37 \dots$. A *recursive pattern* is given by $a_1 = 1$ and $a_n = a_{n-1} + (n - 1)$.

Solution 3: Using an “odd, even, even” pattern, we have $1, 2, 4, 5, 6, 8, 9, 10, 12, \dots$

Solution 4: Skipping all multiples of three, we have $1, 2, 4, 5, 7, 8, 10, 11, 13 \dots$

⋮

We will temporarily restrict ourselves to **arithmetic sequences** and investigate their special properties.

Definition 7.2.13. An **arithmetic sequence** is a sequence in which successive terms differ by a fixed number.

Example 7.2.14. The sequences $1, 4, 7, 10, 13, 17, 21, \dots$ and $2, 6, 10, 14, 18, 22, 26, \dots$ share the trait that the difference between two consecutive terms is a constant. In the first case, each term is three more than the prior term; in the second sequence, each term is four more than its predecessor.

Note that arithmetic sequences are defined recursively by $a_{n+1} = a_n + C$, where C is the constant. Also, since sequences are just functions with the special domain \mathbb{N} , we see that an arithmetic sequence is just a linear function on that domain.

Example 7.2.15. Find a formula for the n th term of an arithmetic sequence with first term 1 and difference 3.

Solution: Here we need to find a formula for the n th term without knowing what natural number n represents. We know that $a_1 = 1$ and $C = 3$, so $a_2 = 1 + 3 = 4$, $a_3 = 4 + 3 = 7$, and so on. Let's put this information into a table and try to organize our data.

n	a_n	
1	1	1
2	$1 + 3$	4
3	$(1 + 3) + 3$	7
4	$(1 + 3 + 3) + 3$	10
5	$(1 + 3 + 3 + 3) + 3$	13

Notice that a_5 is a 1 with four 3's added, a_4 is a 1 with three 3's added, and so on. It looks reasonable that $a_n = 1 + 3(n - 1)$, which simplifies to $a_n = 3n - 2$. Let's check this: $3(5) - 2 = 13$, which is what the fifth term should be.

Theorem 7.2.16. *The n th term for an arithmetic sequence with first term a and difference d is $a_n = a + (n - 1)d$.*

Proof. We defer the proof, by induction on n , until another unit. □

Example 7.2.17. Find the first two terms of an arithmetic sequence if $a_3 = 11$ and $a_{30} = 146$.

Solution: We have that $a_3 = a + (3 - 1)d = 11$ which implies that $a + 2d = 11$. We also have that $a_{30} = a + (30 - 1)d = 146$, hence, $a + 29d = 146$. We now have a system of two equations in two unknowns. This gives $27d = 135 \implies d = 5$, so $a = 1$. Hence, the first two terms are $a_1 = 1$, and $a_2 = 6$.

Definition 7.2.18. A **geometric sequence** is a sequence in which successive terms have a constant ratio r .

These are much like arithmetic sequences, except that instead of *adding* the same number to each term, we *multiply* each term by the same number.

Example 7.2.19. The sequence 1, 2, 4, 8, 16, ... is a geometric sequence with initial term 1 and common ratio 2.

Example 7.2.20. The sequence 2, 6, 18, 54, 162, 468, 1458, ... is a geometric sequence with initial term 2 and common ratio 3.

As was the case with arithmetic sequences, geometric sequences are defined recursively. We will again make a table to try to find a general formula for the n th term of a geometric sequence with initial term a and common ratio r .

n	a_n
1	a
2	ar
3	ar^2
4	ar^3
5	ar^4

We apparently have the formula $a_n = ar^{n-1}$; we will prove that this is correct by induction in the next section.

Theorem 7.2.21. *If a_1, a_2, a_3, \dots is a geometric sequence with common ratio r and initial term $a = a_1$, then $a_n = ar^{n-1}$.*

Example 7.2.22. Above, we had $1 \cdot 2^{n-1}$ and $2 \cdot 3^{n-1}$.

Just as arithmetic sequences were linear functions, geometric sequences are exponential functions and, therefore, have application in growth and decay problems.

Example 7.2.23. Find the eighth term in the geometric sequence $2/3, 2/9, 2/27, \dots$.

Solution: The initial term is $\frac{2}{3}$ and the common ratio is $\frac{1}{3}$, so the eighth term is $\frac{2}{3} \left(\frac{1}{3}\right)^{8-1} = \frac{2}{6561}$.

Exercises

Write the first five terms of each sequence.

1. $\{3n\}$

4. $\left\{\frac{3^n}{n}\right\}$

2. $\{n^2 + 2n\}$

5. $\left\{\frac{3^n}{2n+3}\right\}$

3. $\left\{\left(\frac{4}{3}\right)\right\}$

6. $\left\{\frac{n^2}{2^n}\right\}$

Assuming the given pattern continues, write a formula for the n^{th} term of each sequence suggested by the pattern.

7. $1, 3, 5, 7, 9, 11, \dots$

10. $\frac{1}{2}, 1, 2, 4, 8, \dots$

8. $-3, 1, 5, 9, 13, 17, \dots$

9. $5, 2, -1, -4, -7, -10, \dots$

11. $-3, 6, -12, 24, -48, 96, \dots$

12. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$
13. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$
14. 2, 6, 12, 20, 30, 42, ...
15. 1, -2, 3, -4, 5, -6, ...
16. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \dots$
17. 1, 3, 6, 10, 15, 21, 28, ...

Find the indicated term in each of the given sequences.

18. Find the fifth term of the arithmetic sequence with first term 6 and difference 3.
19. Find the seventh term of the arithmetic sequence with first term -7 and difference 5.
20. Find the fifteenth term of the arithmetic sequence with first term 3 and difference 8.
21. Find the fifth term of the arithmetic sequence with first term 1 and difference π .
22. Find the fifth term of the geometric sequence with first term 2 and ratio 3.
23. Find the sixth term of the geometric sequence with first term $\frac{1}{2}$ and ratio 7.
24. Find the eighth term of the geometric sequence with first term 5 and ratio -3.
25. Find the first term of the arithmetic sequence with seventh term 12 and difference 3.
26. Find the first term of the arithmetic sequence with fifth term -9 and difference -2.
27. Find the first term of the arithmetic sequence with third term 8 and eighth term 23.
28. Find the first term of the arithmetic sequence with eighth term 8 and twentieth term 44.
29. Find the first term of the arithmetic sequence with fifth term -2 and thirteenth term 30.
30. Find x so that x , $x + 2$, and $x + 3$ are terms of a geometric sequence.
31. Find x so that $x - 1$, x , and $x + 2$ are terms of a geometric sequence.
32. Does there exist a sequence $\{a_n\}$ that is both arithmetic and geometric? Why or why not?

7.3 Operations on Functions

Functions give us a new kind of “numbers,” so we once again need to define some notion of equality and introduce some operations.

Definition 7.3.1. Two functions f and g are **equal**, written $f = g$, if and only if

1. f and g have the same domain D , and
2. $f(x) = g(x)$ for all $x \in D$.

Example 7.3.2. Are the functions defined by $f(x) = x + 1$ and $g(x) = \frac{x^2 - 1}{x - 1}$ equal functions?

Solution: No, f and g are not equal because they do not have the same domain. The domain of f is \mathbb{R} , and the domain of g is $\{x \in \mathbb{R} | x \neq 1\}$. However, f and g **do** have the same action on the restricted domain $\{x \in \mathbb{R} | x \neq 1\}$.

When two functions have a common domain, it is sufficient to compare their action on each domain element to determine whether the functions are equal.

Example 7.3.3. The functions $f(x) = x$ and $g(x) = |x|$ have a common domain, \mathbb{R} , but they are not equal functions since they “disagree” on $(-\infty, 0)$. In particular,

$$f(-1) = -1 \neq 1 = g(-1).$$

Example 7.3.4. We have already seen that $f(x) = |x|$ and $g(x) = \sqrt{x^2}$ are equal functions; both have domain \mathbb{R} , and they agree at all elements of their domain.

We will now define some operations on the set of functions and investigate which field properties are satisfied by the operations.

Definition 7.3.5. Let $f : D_1 \rightarrow R_1$ and $g : D_2 \rightarrow R_2$ be functions with R_1 and R_2 subsets of the same field (think of \mathbb{R} and/or \mathbb{C}). The **sum** of f and g is the function $f + g$, given by $(f + g)(x) = f(x) + g(x)$ for $x \in D_1 \cap D_2$.

Example 7.3.6. Let $f(x) = 4x - 6$ and $g(x) = x^2 - 3x + 2$. Then

$$(f + g)(x) = f(x) + g(x) = (4x - 6) + (x^2 - 3x + 2) = x^2 + 4x - 4,$$

and

$$(f + g)\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right) = \left(4 \cdot \frac{1}{2} - 6\right) + \left(\left(\frac{1}{2}\right)^2 - 3 \cdot \frac{1}{2} + 2\right) = -\frac{7}{4}.$$

Naturally, we want to know about field properties. We will have to be careful with the domain, but if we are, what do we get?

Theorem 7.3.7. *Addition of functions is associative and commutative.*

Proof. A function is determined by what it does to its domain. In particular, $f + g$ is determined by the values $(f + g)(x)$, where x is in the domain of $f + g$. Also, notice that $f + g$ and $g + f$ have the same domain since $D_1 \cap D_2 = D_2 \cap D_1$. The key in the proof below is the realization that $f(x)$ and $g(x)$ **are field elements** (members of \mathbb{R}). Consider:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) && \text{Definition of Function Addition} \\ &= g(x) + f(x) && f(x), g(x) \in \mathbb{R} \text{ and addition in a field commutes} \\ &= (g + f)(x) && \text{Definition of Function Addition.}\end{aligned}$$

Therefore, for any x in the domain of $f + g$, $(f + g)(x) = (g + f)(x)$, so $f + g = g + f$.

The proof of associativity is similar and will be left to the reader. \square

Theorem 7.3.8. *Let F be a field. The function $f(x) = 0$ with domain F is an identity for function addition.*

Proof. If h is any function, then $(f + h)(x) = f(x) + h(x) = 0 + h(x) = h(x)$ for all x in the domain of h . Thus $f + h = h$. By commutativity of function addition, $h + f = h$. Therefore, f is an identity for function addition. \square

Theorem 7.3.9. *The function $-f$ given by $(-f)(x) = -(f(x))$ is an additive inverse for the function f .*

Proof. Note that $-f$ has the same domain as f . For any x in the domain of f , we have

$$\begin{aligned}(f + (-f))(x) &= f(x) + (-f)(x) && \text{Definition of Function Addition} \\ &= f(x) + (-(f(x))) && \text{Definition of } (-f)(x) \\ &= 0 && \text{Additive Inverse (remember, } f(x) \in \mathbb{R}\text{!).}\end{aligned}$$

Since we know that addition of functions is commutative, we have $f + (-f) = 0 = (-f) + f$, so $-f$ is the additive inverse of f . \square

Since every function has an additive inverse, we can now define function subtraction.

Definition 7.3.10. Let $f : D_1 \rightarrow R_1$ and $g : D_2 \rightarrow R_2$ be functions with R_1 and R_2 subsets of the same field. The **difference** of f and g is the function $f - g = f + (-g)$, given by $(f - g)(x) = f(x) - g(x)$ for $x \in D_1 \cap D_2$.

Example 7.3.11. Again using $f(x) = 4x - 6$ and $g(x) = x^2 - 3x + 2$, we have

$$(f - g)(x) = (4x - 6) - (x^2 - 3x + 2) = 4x - 6 - x^2 + 3x - 2 = -x^2 + 7x - 8$$

and

$$(f - g)(2) = -(2^2) + 7(2) - 8 = 2.$$

We'll now turn our attention to defining and investigating multiplication of functions.

Definition 7.3.12. Let $f : D_1 \rightarrow R_1$ and $g : D_2 \rightarrow R_2$ be functions with R_1 and R_2 subsets of the same field (again, think of \mathbb{R} and/or \mathbb{C}). The **product** of f and g is the function fg , given by $(fg)(x) = f(x)g(x)$ for all $x \in D_1 \cap D_2$.

Example 7.3.13. Let $f(x) = 4x - 6$ and $g(x) = x^2 - 3x + 2$. Then

$$(fg)(x) = (4x - 6)(x^2 - 3x + 2)$$

and

$$(fg)(-1) = (4(-1) - 6)((-1)^2 - 3(-1) + 2) = (-10)(6) = -60.$$

Theorem 7.3.14. *Multiplication of functions is associative and commutative.*

We omit the proof, as it is very similar to that of the theorem above.

Theorem 7.3.15. *Let F be a field. The function $g(x) = 1$ with domain F is an identity for function multiplication.*

Proof. If h is any function, then $(gh)(x) = g(x)h(x) = 1 \cdot h(x) = h(x)$ for all x in the domain of h . Thus $gh = h$. Since function multiplication commutes, $hg = h$ as well, so $g(x) = 1$ is an identity for function multiplication. \square

Theorem 7.3.16. *Let f be a function with domain D , and let $D' = \{x \in D \mid f(x) \neq 0\}$. Then the function g given by $g(x) = \frac{1}{f(x)}$ is a multiplicative inverse for f on D' .*

Proof. We need to restrict the domain to D' to prevent a division by zero. Now, let $x \in D'$. Then

$$\begin{aligned} (f \cdot g)(x) &= f(x) \cdot g(x) && \text{Definition of function multiplication} \\ &= f(x) \frac{1}{f(x)} && \text{Definition of } g(x). \\ &= 1 && \text{Multiplicative inverses (remember that } f(x) \text{ is a real number!),} \end{aligned}$$

which is the multiplicative identity for functions.

Since we know that multiplication of functions commutes, we also have $f \cdot g = 1 = g \cdot f$, so $g = \frac{1}{f}$ is the multiplicative inverse of f . \square

In practice, the multiplicative inverse of a function is only marginally useful, except for in defining function division. Note: we **do not** use the -1 exponent for denoting the multiplicative inverse of a function.

Definition 7.3.17. Let $f : D_1 \rightarrow R_1$ and $g : D_2 \rightarrow R_2$ be functions with R_1 and R_2 subsets of the same field. The **quotient** of f and g is the function $\frac{f}{g}$, given by $\left(\frac{f}{g}\right)(x) = f(x)/g(x)$ for all $x \in D_1 \cap D_2 \cap \{x \mid g(x) \neq 0\}$ (since division by 0 is not defined).

Example 7.3.18. Let $f(x) = 4x - 6$ and $g(x) = x^2 - 3x + 2$. Then

$$\left(\frac{f}{g}\right)(x) = \frac{4x - 6}{x^2 - 3x + 2}$$

and

$$\left(\frac{f}{g}\right)(0) = \frac{4 \cdot 0 - 6}{0^2 - 3 \cdot 0 + 2} = -3.$$

The domain of $\frac{f}{g}$ is $\{x \in \mathbb{R} | x^2 - 3x + 2 \neq 0\}$. If $x^2 - 3x + 2 = 0$, then $(x - 2)(x - 1) = 0$, so $x = 2$ or $x = 1$ by the Zero Product Theorem. Therefore, the domain of $\frac{f}{g}$ is

$$\mathbb{R} \cap \mathbb{R} \cap \{x \in \mathbb{R} | x \neq 1, 2\} = \{x \in \mathbb{R} | x \neq 1, 2\}.$$

Theorem 7.3.19. *Function multiplication distributes over function addition.*

Proof. Let f, g , and h be functions with domains A, B , and C , respectively. Let $D = A \cap B \cap C$, so that D is the domain of both $f \cdot (g + h)$ and $fg + fh$. (Note that $f \cdot (g + h)$ does NOT refer to function evaluation in this context.) Let $x \in D$. Then

$$\begin{aligned} (f \cdot (g + h))(x) &= f(x)((g + h)(x)) && \text{Definition of Function Multiplication} \\ &= f(x)(g(x) + h(x)) && \text{Definition of Function Addition} \\ &= f(x)g(x) + f(x)h(x) && \text{Distribution of multiplication over addition in } \mathbb{R} \\ &= (fg)(x) + (fh)(x) && \text{Definition of Function Multiplication} \\ &= (fg + fh)(x) && \text{Definition of Function Addition.} \end{aligned}$$

Therefore $f(g + h)$ and $fg + fh$ have the same values for all $x \in D$, so they must be the same function. That is, $f \cdot (g + h) = fg + fh$. Similarly, $(g + h) \cdot f = gf + hf$ (or use commutativity of function multiplication). \square

Therefore, as long as we are careful with domains, we have that functions satisfy the field axioms, so we may use the theorems we proved for fields in general. However, we really must pay attention to domains to avoid inappropriately using field theorems.

We have one more operation to define.

Definition 7.3.20. Let $f : D_1 \rightarrow R_1$ and $g : D_2 \rightarrow R_2$ be functions with R_1 and R_2 subsets of the same field. The **composition** of f by g is $f \circ g$, given by $(f \circ g)(x) = f(g(x))$. The domain of the composition is $\{x \in D_2 | g(x) \in D_1\}$.

Think of composition of functions as a two-step process. First, evaluate g at x . Then take the *output* from that, and substitute it into f . This is why the domain looks like it does: in evaluating $f \circ g$ at x , the first step is to evaluate g at x , so x had better be in D_2 , the domain of g . The second step is to evaluate f at the *output* from g , so that output (which is $g(x)$) had better be in D_1 , the domain of f .

Example 7.3.21. Let $f(x) = \sqrt{x}$ and $g(x) = x - 4$. The domain of g is \mathbb{R} , and the domain of f is $\{x \in \mathbb{R} | x \geq 0\}$. The domain of $f \circ g$ is therefore $\{x \in \mathbb{R} | x - 4 \geq 0\} = \{x \in \mathbb{R} | x \geq 4\}$.

$$(f \circ g)(x) = \sqrt{x - 4}.$$

Example 7.3.22. It is interesting that composition of functions does NOT commute. Consider $f(x) = x^2$ and $g(x) = x + 1$. $(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$. On the other hand, $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$, which is not at all the same function!

Theorem 7.3.23. *Composition of functions is associative.*

We omit the proof of this theorem; it is relatively straight-forward as long as one is careful about the technicalities involved with the necessary domain restriction.

Theorem 7.3.24. *Let F be a field. The function $\iota(x) = x$ with domain F is an identity for composition.*

Note: ι is the Greek letter lower-case iota.

Proof. Let f be a function with domain $D \subseteq F$, and let $x \in D$. Then $(f \circ \iota)(x) = f(\iota(x)) = f(x)$, and $(\iota \circ f)(x) = \iota(f(x)) = f(x)$. Thus $f \circ \iota = f = \iota \circ f$, so ι is an identity for composition. \square

At this point, one naturally wonders about the existence of inverses with respect to composition. Like multiplicative inverses for matrices, inverse functions (with respect to composition) do not always exist. We explore inverse functions in detail in a later section.

We end this section with a definition that will help us analyze graphical symmetry in the next section.

Definition 7.3.25. An **even function** is a function f that satisfies $f(-x) = f(x)$ for all x in the domain of f . An **odd function** is a function that satisfies $f(-x) = -f(x)$ for all x in the domain of f .

Example 7.3.26. The function defined by $f(x) = x^2 + 1$ is an even function because

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x).$$

The function defined by $g(x) = -x^3 + x$ is an odd function because

$$g(-x) = -(-x)^3 + (-x) = x^3 - x = -(-x^3 + x) = -g(x).$$

The function defined by $h(x) = x^2 + x$ is neither even nor odd since

$$h(-x) = (-x)^2 + (-x) = x^2 - x,$$

which equals neither $h(x)$ nor $-h(x)$.

Exercises

Perform the indicated operations and simplify. Find the domain for each function, as well. (Remember that the domain of sum, difference, etc. must be found *before* simplifying.)

1. For $f(x) = \frac{1}{x^2 + 1}$ and $g(x) = \frac{1}{x - 1}$, find $f + g$, $\frac{f}{g}$, and $(f - g)(4)$.
2. Let $f(x) = x^2 - 3x + 5$, $g(x) = 3x + 4$, and $h(x) = -2x^2 - 9$. Find $f + g + h$ and $\frac{f - h}{g}$.
3. For $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find fg and $f \circ g$.
4. **Profit** is defined as revenue (income) minus cost (expense). If a certain company's revenue function is described by $R(x) = -x^2 + 18x + 6$ and its cost function is $C(x) = 4x - 22$, find its profit function $P(x)$. (Here x is the number of units produced.)

We can define a function from the set \mathbb{R}^2 of 2×1 column matrices to itself by matrix multiplication. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f(X) = AX$.

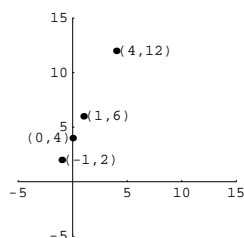
5. Verify that if X is 2×1 , then $f(X)$ is 2×1 .
6. Compute $f\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$.
7. Compute $f\left(\begin{bmatrix} 12 \\ -8 \end{bmatrix}\right)$.
8. Show that f is a **linear operator**; that is, show that $f(X + Y) = f(X) + f(Y)$. (This makes this particular function extremely special; most functions do not behave this way – think about x^2 , for example, or $x - 3$.)
9. Let $f(x) = \frac{1}{x - 1}$ and $g(x) = \sqrt{x}$.
 - (a) Find $(f \circ g)(x)$ (including its domain).
 - (b) Find $(g \circ f)(x)$ (including its domain).
 - (c) What is $(f \circ g)(4)$?
 - (d) What is $(g \circ f)(3)$?
10. Let $f(x) = x^2$ and $g(x) = 2x - 1$.
 - (a) Find $(f \circ g)(x)$ (including its domain).
 - (b) Find $(g \circ f)(x)$ (including its domain).
 - (c) What is $(f \circ g)(-2)$?
 - (d) What is $(g \circ f)(-2)$?

Determine whether each function is even, odd, or neither.

26. Prove that if f is an odd function, then so are $\frac{1}{f}$ and $-f$.
27. Prove that the product of an even function and an odd function is odd.
28. Prove that addition of functions is associative.
29. Explain the difference between f and $f(x)$, where f is a function.
30. Prove that multiplication of functions is associative.
31. Prove that multiplication of functions is commutative.

7.4 Graphing Functions

We return to graphing. The power of graphing is in the visual representation it can give of an otherwise mysterious relationship. Given the points $(-1, 2)$, $(4, 12)$, $(0, 4)$, $(1, 5)$, can you see the relationship among them? Probably not! What if we graph them?



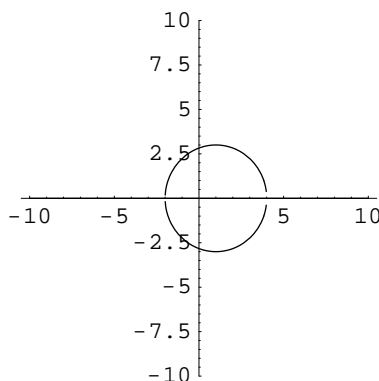
Viewing the graph, it is not hard to believe that these points all lie on a line, although that was not readily apparent from just looking at the ordered pairs. Scientists, business people, journalists, and, in fact, most professions regularly use graphical representations of data to help determine or illustrate relationships.

In this section we will build up a library of basic graphs we should be able to recognize, and then see how they may be transformed. You will want to memorize the graphs of the basic functions so that you can recognize them at a glance.

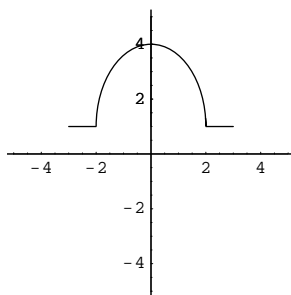
First, let's consider the conditions under which a given graph is the graph of a function $y = f(x)$. Recall that for a relation to be a function we need each x -value in the domain to be paired with exactly one y -value. Since equal x -values correspond graphically to points on a vertical line (consider the graph of $x = 2$), we have the following theorem.

Theorem 7.4.1 (Vertical Line Test). *A graph in the xy -coordinate plane is the graph of a function $y = f(x)$ if and only if every vertical line intersects the graph at most once.*

Example 7.4.2. The relationship defined by the equation $(x - 1)^2 + y^2 = 9$ does not represent a function because the graph of this equation is a circle with center $(1, 0)$ and radius 3, which fails the vertical line test.



Example 7.4.3. Does the following graphical relationship represent a function? If so, find the domain and range of the function.

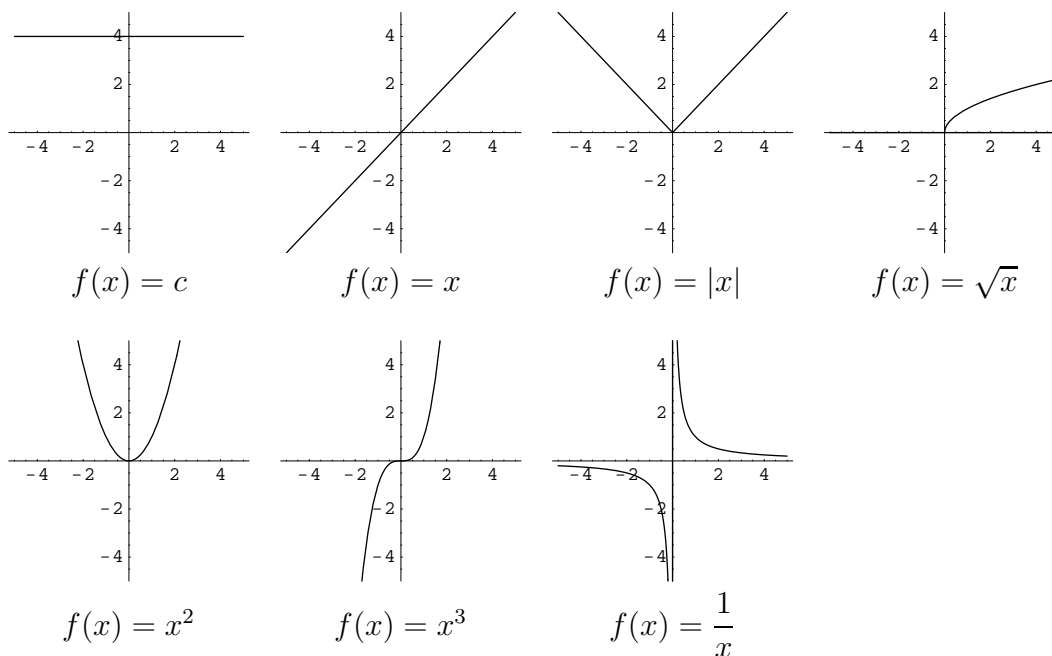


Solution: Since the graph passes the vertical line test, this relationship is a function, f . The domain is the set of x -values used as first coordinates and the range is the set of second coordinates of the ordered pairs comprising f . Hence, the domain appears to be $[-3, 3]$, and the range is $[1, 4]$.

For the vertical line test to be very useful to us, we need to have knowledge of the graph of a given relation. For a completely unknown function, we must plot enough ordered pairs belonging to the function to give us a general idea of what its graph looks like. Consider the following “basic” functions.

$f(x)$	Name	Returns...	Example	Domain
c	Constant	c regardless of x .	$f(-2) = c$	\mathbb{R}
x	Identity	the input.	$f(-2) = -2$	\mathbb{R}
$ x $	Absolute Value	x if $x \geq 0$, and $-x$ if $x < 0$.	$f(-2) = 2$	\mathbb{R}
\sqrt{x}	Square root	$a \geq 0$ such that $a^2 = x$	$f(4) = 2$	$[0, \infty)$
x^2	Square	the square of x	$f(-2) = 4$	\mathbb{R}
x^3	Cube	the cube of x	$f(-2) = -8$	\mathbb{R}
$\frac{1}{x}$	Reciprocal	the multiplicative inverse of x	$f(2/5) = 5/2$	$\{x \in \mathbb{R} x \neq 0\}$

The graphs of these “basic” functions are given below.



We have several transformations we can do with these (and other) functions and still get functions of the same basic shape. Let $c > 0$, and let f be a given function (not necessarily one of those above). We will compare the graph of $f(x)$ to the graph of $g(x) = f(x + c)$.

Remember that in graphing a function, the x -coordinate represents a domain element and the y -coordinate (or “height”) represents the function value at x . Notice that, for example, $f(0) = g(-c)$, so g has the same height at $-c$ as f does at 0. Also, $f(-2) = g(-2 - c)$, so g has the same height at $-2 - c$ as f does at -2 . In general, g will have the same height at $x - c$ as f does at x . Geometrically, this just means that the graph of g follows *exactly the same* “ups and downs” as f does; that is, the graph of g has the same shape as the graph of f . However, while the graph of g attains the same heights as does the graph of f , it does so at $x - c$ instead of at x ; that is, c units to the left of where f does.

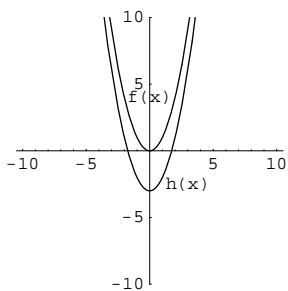
In summary, the graph of $g(x) = f(x + c)$ is exactly the same shape as the graph of f , but translated to the left by c units. Similar arguments lead to the table below.

New function:	Effect on graph of $y = f(x)$
$h(x) = f(x) + c$	Vertical shift up c units
$h(x) = f(x) - c$	Vertical shift down c units
$h(x) = f(x + c)$	Horizontal shift left c units
$h(x) = f(x - c)$	Horizontal shift right c units
$h(x) = -f(x)$	Reflection across x -axis
$h(x) = f(-x)$	Reflection across y -axis
$h(x) = cf(x)$	Vertical stretch by a factor of c for $c > 1$; shrink for $0 < c < 1$
$h(x) = f(cx)$	Horizontal shrink by a factor of c for $c > 1$; stretch for $0 < c < 1$

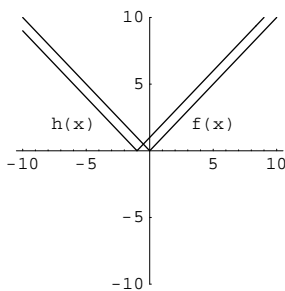
One can see that there are two fundamentally different ways to modify a graph: one can modify the input into f , as $f(x + c)$ or $f(cx)$, or one can modify the output from f , as $f(x) + c$ or $cf(x)$. **The parentheses matter**; read very carefully and consider every symbol.

Example 7.4.4. For each function, identify the basic function and the transformation involved.

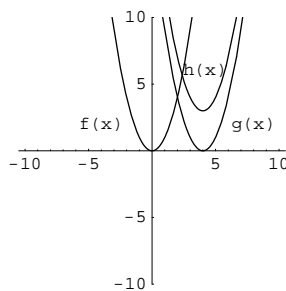
1. $h(x) = x^2 - 3$. The basic function is $f(x) = x^2$; h is a downward shift by 3 units.
2. $h(x) = |x + 1|$. The basic function is $f(x) = |x|$; h is a shift to the left by 1 unit.
3. $h(x) = (x - 4)^2 + 3$. The basic function is $f(x) = x^2$; we have two transformations here. If we put $g(x) = (x - 4)^2$, then $h(x) = g(x) + 3$. To obtain the graph of g , shift the graph of f right 4 units. To obtain the graph of h , shift the graph of g up 3 units.
4. $h(x) = -x^2$. The basic function is $f(x) = x^2$; h is a reflection across the x -axis.
5. $h(x) = 7x^2$. The basic function is $f(x) = x^2$; h is a vertical stretch by a factor of 7.
6. $h(x) = \frac{x^2}{4}$. The basic function is $f(x) = x^2$; h is a vertical shrink by a factor of $1/4$.
7. $h(x) = \sqrt{-x}$. The basic function is $f(x) = \sqrt{x}$; h is a reflection across the y -axis.
8. $h(x) = \sqrt{2x}$. The basic function is $f(x) = \sqrt{x}$. $h(x) = f(2x)$, so its graph is a horizontal compression by a factor of 2 of that of f .
9. $h(x) = -|x| + 3$. The basic function is $f(x) = |x|$. Let $g(x) = -f(x)$, so $h(x) = g(x) + 3$. Therefore, we first reflect the graph of f across the x -axis, and then shift up 3 units.
10. $h(x) = \sqrt{-x + 7} = \sqrt{-(x - 7)}$. The basic function is $f(x) = \sqrt{x}$. Let $g(x) = \sqrt{-x} = f(-x)$. Then the graph of g is the reflection of the graph of f across the y -axis. Also, $h(x) = g(x - 7) = f(-(x - 7)) = \sqrt{-(x - 7)} = \sqrt{-x + 7}$. Therefore, the graph of h is the graph of g shifted right 7 units. We again had to perform two transformations.
11. $h(x) = \frac{-1}{x + 2}$. The basic function is $f(x) = \frac{1}{x}$. Let $g(x) = -f(x)$ (a reflection across the x -axis), and then $h(x) = g(x + 2)$ (a shift right by 2 units).
12. $h(x) = (2x + 6)^3$. The basic function is $f(x) = x^3$. Let $g(x) = f(2x)$. Then $h(x) = g(x + 3)$. Thus the graph of h is the graph of x^3 after a horizontal compression by a factor of 2 and a shift to the left 3 units. We could also have used $g(x) = f(x + 6)$, a shift to the left by 6 units, followed by $h(x) = g(2x) = f(2x + 6)$, a horizontal compression by a factor of 2. We can choose either way, but we must be careful not to mix them together.



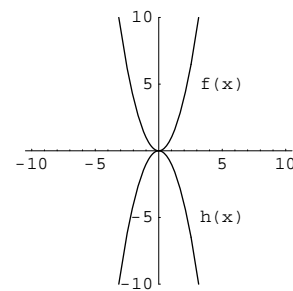
1. $h(x) = x^2 - 3$



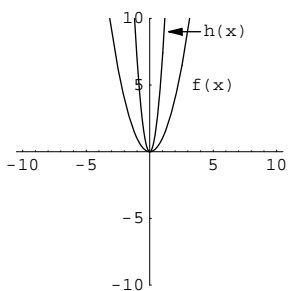
2. $h(x) = |x + 1|$



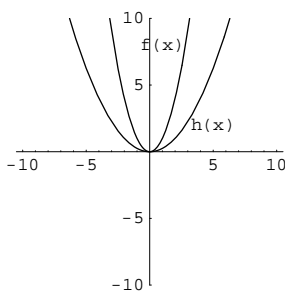
3. $h(x) = (x - 4)^2 + 3$



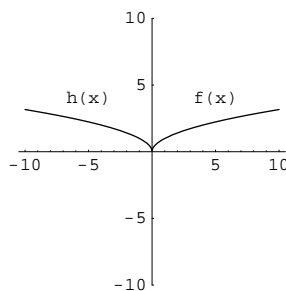
4. $h(x) = -x^2$



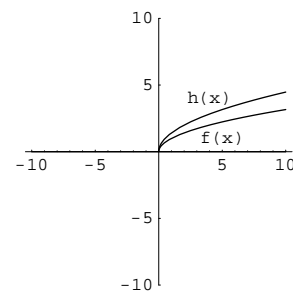
5. $h(x) = 7x^2$



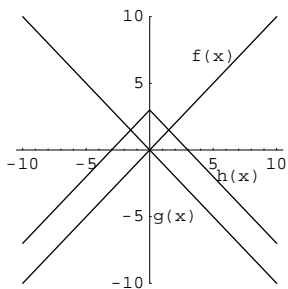
6. $h(x) = \frac{1}{4}x^2$



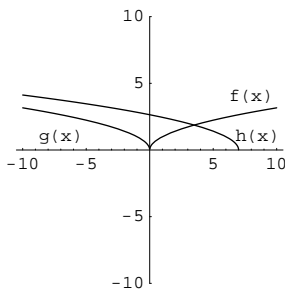
7. $h(x) = \sqrt{-x}$



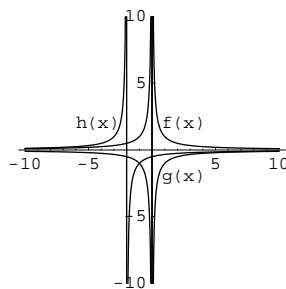
8. $h(x) = \sqrt{2x}$



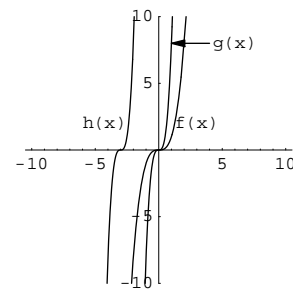
9. $h(x) = -|x| + 3$



10. $h(x) = \sqrt{-x + 7}$



11. $h(x) = \frac{-1}{x + 2}$



12. $h(x) = (2x + 4)^3$

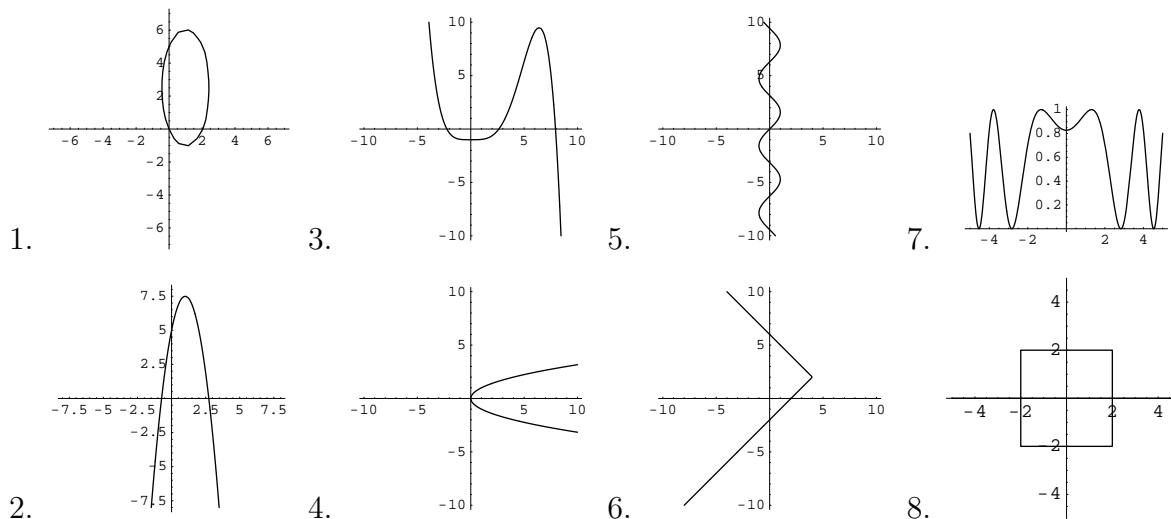
Note that the graph touches the x -axis if and only if the y -coordinate is 0. The corresponding x -value is called a **zero** of f .

Definition 7.4.5. If $f(x) = 0$, then x is a **zero** of f .

Zeros are especially important in equation solving, one of the major purposes of algebra.

Exercises

Determine whether the given graph is the graph of a function.



9. For each graph in problems 1 through 8 that represents a function, determine the domain and range from the graph as well as possible. (You may need to extrapolate beyond what you can see on the graph. Realize, though, that doing so is somewhat hazardous!)
10. For each graph in problem 1 that represents a function f , determine $f(2)$ from the graph, if possible. Realize that this will only be an approximation. If it is not possible, explain why not.

Sketch the graph of each function. If possible, do so by transforming the graph of a known function rather than plotting points.

11. $f(x) = x^2 + 1$

14. $f(x) = \sqrt{2x - 5}$

17. $f(x) = \sqrt{-(x + 1)}$

12. $f(x) = \frac{1}{x - 2}$

15. $f(x) = 2|x + 3| - 1$

18. $f(x) = (3x)^3$

13. $f(x) = \frac{x + 1}{x}$

16. $f(x) = -|x - 2|$

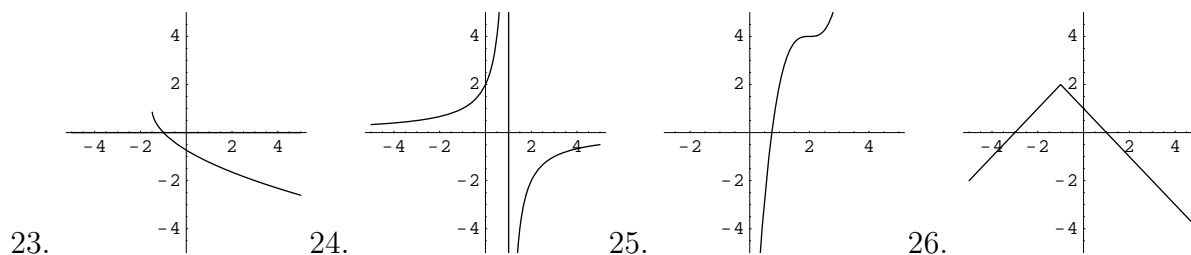
19. $f(x) = -\frac{1}{x - 3} + 2$

20. Convince yourself that each of the transformations in the table “Transformations of Graphs” does what the table claims it will do.

21. The **floor** function is another basic function; $\text{floor}(x)$, also written $\lfloor x \rfloor$ (pronounced “floor x ”) is the greatest integer that is less than or equal to x . For example, $\lfloor 2.75 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, $\lfloor -4.1 \rfloor = -5$, and $\lfloor 461 \rfloor = 461$. Sketch a graph of the floor function.

22. The **ceiling** function is another basic function; $\text{ceil}(x)$, also written $\lceil x \rceil$ (pronounced “ceiling x ”) is the least integer that is greater than or equal to x . For example, $\lceil 2.75 \rceil = 3$, $\lceil \pi \rceil = 4$, $\lceil -4.1 \rceil = -4$, and $\lceil 461 \rceil = 461$. Sketch a graph of the ceiling function.

Each graph below is a transformation of some basic function. Identify the basic function.



27. For each $x \in \mathbb{R} - \mathbb{Z}$, show that $\lceil x \rceil = \lfloor x + 1 \rfloor$.

28. The following data was collected in an experiment in which the height h of a dropped ball was measured (in centimeters) at several times t . Sketch a graph of the height h versus time t , and decide whether the graph is a transformation of one of our basic graphs.

$h(t)$	202	201	200	196	191	185	178	169	159	148	135	122	107	90	73	54	34	12
t	$\frac{3}{60}$	$\frac{5}{60}$	$\frac{7}{60}$	$\frac{9}{60}$	$\frac{11}{60}$	$\frac{13}{60}$	$\frac{15}{60}$	$\frac{17}{60}$	$\frac{19}{60}$	$\frac{21}{60}$	$\frac{23}{60}$	$\frac{25}{60}$	$\frac{27}{60}$	$\frac{29}{60}$	$\frac{31}{60}$	$\frac{33}{60}$	$\frac{35}{60}$	$\frac{37}{60}$

7.5 Inverse Functions

We now return to the notion of inverse functions (with respect to function composition). Recalling the definition of an inverse element, the definition of composition, and the fact that the identity function for composition is given by $\iota(x) = x$, we have the following definition.

Definition 7.5.1. If f and g are two functions satisfying $f \circ g = \iota$ and $g \circ f = \iota$, then f and g are called **inverses** of each other. In this case, f and g are said to each have an inverse, f is the inverse of g , and g is the inverse of f . These relationships are denoted by $f = g^{-1}$ and $g = f^{-1}$, respectively.

NOTE: The f^{-1} notation refers to the inverse of f under **composition**, *not* the multiplicative inverse of f . This similarity in notation is unfortunate, but standard. It should be clear from the context whether we are talking about an inverse function or a multiplicative inverse of a number.

Since two functions are equal provided they have identical domains and the same action on each element of their domain, we have the following equivalent statement.

Theorem 7.5.2. If f and g are two functions satisfying $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f , then f and g are called **inverses** of each other.

Example 7.5.3. Verify that the functions $f(x) = 2x - 1$ and $g(x) = \frac{x+1}{2}$ are inverse functions.

Solution: First note that f and g have the same domain, namely \mathbb{R} . So, we compute the compositions $f \circ g(x)$ and $g \circ f(x)$ for all $x \in \mathbb{R}$ and see that we get x .

$$\begin{array}{l|l}
 (f \circ g)(x) & (g \circ f)(x) \\
 = f(g(x)) & = g(f(x)) \\
 = f\left(\frac{x+1}{2}\right) & = g(2x-1) \\
 = 2\left(\frac{x+1}{2}\right) - 1 & = \frac{(2x-1)+1}{2} \\
 = (x+1) - 1 & = \frac{2x}{2} \\
 = x & = x
 \end{array}$$

Example 7.5.4. Verify that the functions $f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1}{x} - 1$ are inverse functions.

Solution: We need to compute $f \circ g(x)$ for all $x \neq 0$ and $g \circ f(x)$ for all $x \neq -1$ and see that these both equal x .

$$\begin{array}{l|l}
 f \circ g(x) & = f(g(x)) \\
 & = f\left(\frac{1}{x} - 1\right) \\
 & = \frac{1}{1 + \left(\frac{1}{x} - 1\right)} \\
 & = \frac{1}{(1/x)} \\
 & = x \\
 \hline
 g \circ f(x) & = g(f(x)) \\
 & = g\left(\frac{1}{1+x}\right) \\
 & = \frac{1}{\left(\frac{1}{1+x}\right)} - 1 \\
 & = (1+x) - 1 \\
 & = x
 \end{array}$$

Suppose the functions f and g are functions defined as sets of ordered pairs. Say that $f = \{(x, y) | y = f(x)\}$ and $g = \{(x', y') | y' = g(x')\}$. Then f and g are inverse functions if and only if $f(g(x')) = x'$ for all x' (in the domain of g) and $g(f(x)) = x$ for all x (in the domain of f). If $y = f(x)$, then $g(f(x)) = g(y) = x$. Similarly, if $y' = g(x')$, then $f(g(x')) = f(y') = x'$. Thus we have that inverse functions, as sets of ordered pairs, have “reversed” coordinates.

Theorem 7.5.5. *Let f and g be inverse functions. Then $y = f(x)$ if and only if $x = g(y)$. That is, $(x, y) \in f$ if and only if $(y, x) \in g$.*

It may be seen by direct substitution that if the above relationship holds, then f and g are indeed inverse functions. (This is what we did above.) Thus, for the function f to have an inverse *function*, it must be that the relation obtained by reversing all the ordered pairs of f is also a function.

That is, the function $f = \{(x, y) | y = f(x)\}$ has an inverse g if and only if the relation $g = \{(y, x) | y = f(x)\}$ is also a function. In effect, we need each y to be “used” exactly once. A function satisfying this new condition is called a **one-to-one** function.

Definition 7.5.6. A function f is called **one-to-one** provided whenever $f(a) = f(b)$, $a = b$. In other words, no element of the range is paired with more than one element of the domain.

Suppose that $c = f(a) = f(b)$. In terms of ordered pairs, the definition of one-to-one means that if (a, c) and (b, c) are both elements of f , then (c, a) and (c, b) can only be elements of another function if $a = b$; otherwise, c would be paired with *two* elements of the range. To verify that a given function is one-to-one, we prove the conditional: if $f(a) = f(b)$, then $a = b$.

Example 7.5.7. Verify that the function given by $f(x) = \sqrt{x}$ is one-to-one.

Solution: We prove the conditional: if $f(a) = f(b)$, then $a = b$.

$$\begin{array}{ll}
 f(a) & = f(b) & \text{Hypothesis} \\
 \sqrt{a} & = \sqrt{b} & \text{Definition of } f \\
 (\sqrt{a})^2 & = (\sqrt{b})^2 & \text{_____} \\
 a & = b & \text{_____}
 \end{array}$$

Therefore, $f(x) = \sqrt{x}$ is one-to-one.

Example 7.5.8. Verify that the function given by $g(x) = x^2$ is NOT one-to-one.

Solution: We show the conditional, “If $g(a) = g(b)$, then $a = b$ ” is false. To do this, we show that even when the hypothesis is true, the conclusion might be false (either algebraically or by a counter-example).

$$\begin{array}{ll}
 g(a) = g(b) & \text{Hypothesis} \\
 a^2 = b^2 & \text{Definition of } f \\
 \sqrt{a^2} = \sqrt{b^2} & \text{_____} \\
 |a| = |b| & \text{_____} \\
 a = \pm b & \text{_____}
 \end{array}$$

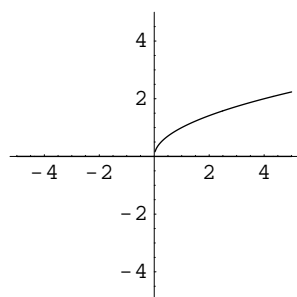
Therefore, $g(x) = x^2$ is NOT one-to-one.

Alternatively, a counterexample where $g(a) = g(b)$ but $a \neq b$ will suffice. Take, for instance, $a = 2$ and $b = -2$.

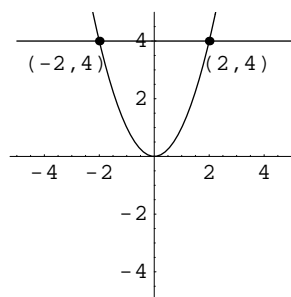
The graph of a function can also be used to determine whether a function is one-to-one. A function is one-to-one provided each y -value “gets used” at most once. Since y -values correspond graphically to horizontal lines, we have the following theorem.

Theorem 7.5.9 (Horizontal Line Test). *A function is one-to-one if and only if every horizontal line intersects its graph in the xy -coordinate plane at most once.*

Example 7.5.10. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = x^2$ from the examples above. They are graphed below. Notice that the horizontal line $y = 4$ intersects g in both points $(-2, 4)$ and $(2, 4)$, but no horizontal line will intersect the graph of f more than once.



$$f(x) = \sqrt{x}$$



$$g(x) = x^2$$

To formally reiterate the relationship between inverse functions and one-to-one functions, we have the following theorem.

Theorem 7.5.11. *A function f has an inverse function if and only if f is one-to-one.*

Note that it is possible to define an inverse for a function that is NOT one-to-one by restricting the domain so that the function is one-to-one on the new domain.

Example 7.5.12. The function $f(x) = x^2$ is one-to-one on $[0, \infty)$ and has inverse $f^{-1}(x) = \sqrt{x}$.

Recall that the graph of f looks like $\{(x, y) | y = f(x)\}$, so the graph of f^{-1} (if it exists) looks like $\{(y, x) | x = f^{-1}(y)\}$. The effect of this on the graph is to reflect the graph of f across the line $y = x$. Notice that reflecting this way will turn a horizontal line (which we used to perform the horizontal line test) into a vertical line (which we use to perform the vertical line test for functions). This confirms the theorem above; a function will have an inverse if and only if it is one-to-one.

The inverse of a given function f may be found algebraically via the following procedure.

1. Verify that f is one-to-one.
2. Find a rule for $f^{-1}(x)$.
 - (a) Write $y = f(x)$. (This is the rule for f .)
 - (b) Interchange x and y in the equation. (This creates an ordered pair (y, x) that also belongs to f since we have $x = f(y)$.)
 - (c) Solve for y . (This gives $y = f^{-1}(x)$, so the ordered pair (x, y) belongs to f^{-1} .)
3. Verify that f and f^{-1} are indeed inverse functions.

Example 7.5.13. Find the inverse of $f(x) = 3x - 2$, provided the inverse exists.

1. Verify that f is one-to-one:

$$\begin{array}{lll}
 f(a) & = & f(b) \quad \text{Hypothesis} \\
 3a - 2 & = & 3b - 2 \quad \text{Definition of } f \\
 3a & = & 3b \quad \text{Additive Cancellation} \\
 a & = & b \quad \text{Multiplicative Cancellation}
 \end{array}$$

2. Find a rule for f^{-1} :

$$\begin{array}{lll}
 y & = & f(x) \quad \text{Write } y = f(x) \\
 y & = & 3x - 2 \quad \text{Function evaluation} \\
 x & = & 3y - 2 \quad \text{Interchange } x \text{ and } y \\
 y & = & \frac{x + 2}{3} \quad \text{Solve for } y \\
 f^{-1}(x) & = & \frac{x + 2}{3} \quad y = f^{-1}(x)
 \end{array}$$

3. Verify that f and f^{-1} are inverse functions:

$$\begin{array}{l|l}
 f \circ f^{-1}(x) = f(f^{-1}(x)) & f^{-1} \circ f(x) = f^{-1}(f(x)) \\
 = f\left(\frac{x+2}{3}\right) & = f^{-1}(3x-2) \\
 = 3\left(\frac{x+2}{3}\right) - 2 & = \frac{(3x-2)+2}{3} \\
 = (x+2) - 2 & = \frac{3x}{3} \\
 = x & = x
 \end{array}$$

We provide a slightly more complicated example for illustration purposes. The process is identical to that in the example above, only the specific calculations are different.

Example 7.5.14. Find the inverse of $f(x) = \frac{5x}{4x+1}$, provided the inverse exists.

1. Verify that f is one-to-one:

$$\begin{array}{ll}
 f(a) = f(b) & \text{Hypothesis} \\
 \frac{5a}{4a+1} = \frac{5b}{4b+1} & \text{Definition of } f \\
 5a(4b+1) = 5b(4a+1) & \text{Fraction equivalence} \\
 a(4b+1) = b(4a+1) & \text{Multiplicative Cancellation} \\
 4ab+a = 4ab+b & \text{Distributive Law} \\
 a = b & \text{Additive Cancellation}
 \end{array}$$

2. Find a rule for f^{-1} :

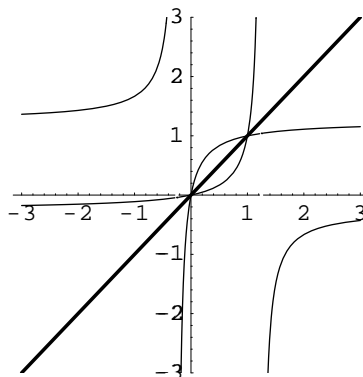
$$\begin{array}{ll}
 y = f(x) & \text{Write } y = f(x) \\
 y = \frac{5x}{4x+1} & \text{Function evaluation} \\
 x = \frac{5y}{4y+1} & \text{Interchange } x \text{ and } y \\
 y = \frac{-x}{4x-5} & \text{Solve for } y \\
 f^{-1}(x) = \frac{-x}{4x-5} & y = f^{-1}(x)
 \end{array}$$

3. Verify that f and f^{-1} are inverse functions:

$$\begin{array}{l|l}
 f \circ f^{-1}(x) = f(f^{-1}(x)) & f^{-1} \circ f(x) = f^{-1}(f(x)) \\
 = f\left(\frac{-x}{4x-5}\right) & = f^{-1}\left(\frac{5x}{4x+1}\right) \\
 = \frac{5\left(\frac{-x}{4x-5}\right)}{4\left(\frac{-x}{4x-5}\right)+1} & = \frac{-\left(\frac{5x}{4x+1}\right)}{4\left(\frac{5x}{4x+1}\right)-5} \\
 = \frac{\left(\frac{-5x}{4x-5}\right)}{\left(\frac{-4x+(4x-5)}{4x-5}\right)} & = \frac{\left(\frac{-5x}{4x+1}\right)}{\left(\frac{20x-5(4x+1)}{4x+1}\right)} \\
 = \frac{\left(\frac{-5x}{4x-5}\right)}{\left(\frac{-5}{4x-5}\right)} & = \frac{\left(\frac{-5x}{4x+1}\right)}{\left(\frac{-5}{4x+1}\right)} \\
 = x & = x
 \end{array}$$

How do the graphs of a one-to-one function and its inverse compare? When we interchange x and y , the effect on the graph is to reflect it across the line $y = x$. We do not prove this fact here; it is a topic more for a course in analytic geometry. Instead, we will illustrate it with an example.

Example 7.5.15. Consider the function $f(x) = \frac{5x}{4x+1}$ from the previous example, with inverse $f^{-1}(x) = \frac{-x}{4x-5}$. The graphs are below, graphed with the line $y = x$.



Notice that if f and g are inverses, then the domain of f is the range of g , and the range of f is the domain of g .

We have used the word “inverse” in several contexts. Originally, we spoke only of additive and multiplicative inverses, and we were very careful about saying which we were using. In some contexts, however, we just say “inverse” without specifying what kind. When we say “the” inverse of a matrix M , we mean the *multiplicative* inverse of M every time. When we say “the” inverse of a function f , we always mean the inverse of f with respect to *composition*. You will need to be alert for context so you can tell what is meant by “the inverse” of some object.

Exercises

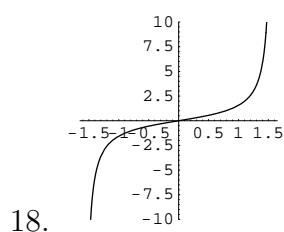
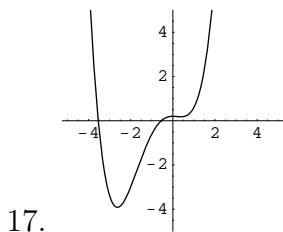
Verify that the given functions are inverses, or show that they are not.

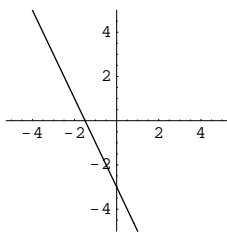
- $f(x) = \sqrt{2x+1} - 3$ for $x \in [-1/2, \infty)$ and $g(x) = \frac{(x+3)^2 - 1}{2}$ for $x \in [-3, \infty)$.
- $f(x) = \frac{3x+1}{x-1}$ and $g(x) = \frac{x+1}{x-3}$
- $f(x) = x - 5$ and $g(x) = x + 5$.
- Determine which “basic” functions (from the section on graphing) are one-to-one.

Determine whether the given function is one-to-one. If a function is one-to-one, find its inverse.

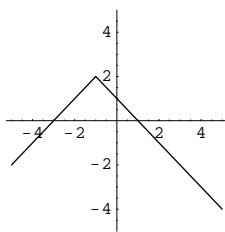
- | | |
|---|--------------------------------|
| 5. $f(x) = (x+2)^3$ | 12. $r(x) = \frac{2}{x+1}$ |
| 6. $g(x) = (x+3)^2$ | 13. $g(x) = \frac{-5}{2x-3}$ |
| 7. $f = \{(1, 3), (2, 5), (0, 4), (-2, 2), (5, 5)\}$ | 14. $h(x) = \frac{2x}{3x+4}$ |
| 8. $h(t) = -16t^2 + 80t + 100$ | 15. $f(x) = \frac{3x+5}{6x-1}$ |
| 9. $g = \{(1, 2), (2, 3), (0, 4), (-2, -3), (5, 5)\}$ | 16. $r(x) = \frac{2x+7}{3x-2}$ |
| 10. $h(x) = \frac{x^2-1}{x+1}$ | |
| 11. $f(x) = \frac{ x }{x}$ | |

Determine which functions are one-to-one. If a function is one-to-one, find its inverse graphically.





19.



20.

21. Explain the difference between the idea of a function and the idea of a one-to-one function.
22. In the previous section, we gave data associated with dropping a ball. The function

$$h(t) = -490.5t^2 + 203$$

gives an approximation for the height of this ball after t seconds.

- Verify that this function is one-to-one on the interval $[0, \infty)$.
- Find an inverse for h on the interval $[0, \infty)$.
- What does the inverse function tell you?

