Chapter 8

Exponential and Logarithmic Functions

This unit defines and investigates exponential and logarithmic functions. We motivate exponential functions by their "similarity" to monomials as well as their wide variety of application. We introduce logarithmic functions as the inverse functions of exponential functions and exploit our previous knowledge of inverse functions to investigate these functions. In particular, we use this inverse relationship for the purpose of solving exponential and logarithmic equations

Objectives

- To define exponential and logarithmic functions
- To investigate the properties of exponential and logarithmic functions
- To introduce some applications of exponential and logarithmic functions
- To solve exponential and logarithmic equations

Terms

- exponential function
- logarithmic function

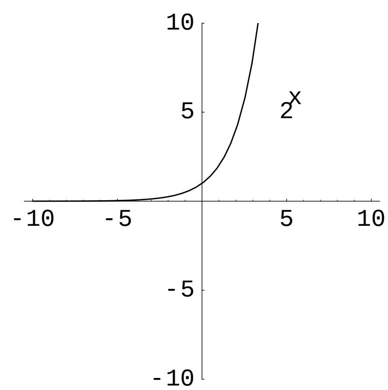
8.1 Exponential Functions

Up to this point, we have been concerned solely with polynomial functions like $f(x) = x^3$ which have a constant exponent and variable base. We now want to consider functions like 3^x which have a variable exponent and constant base. What makes such functions important is the wide variety of applications they have: compound interest, radioactive decay, learning curves, charge in a capacitor, etc.

Definition 8.1.1. The **exponential function** f with **base** a is $f(x) = a^x$, where $a > 0$, $a \neq 1$, $x \in \mathbb{R}$. The domain of f is $\mathbb{R}$, and the range is $(0, \infty)$. (0 itself is not in the range of f.)

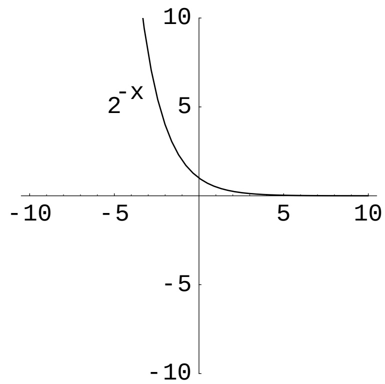
Example 8.1.2. Consider $f(x) = 2^x$. We will create a table of values for x and $f(x)$, and then sketch a graph of f.

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Observe that f is increasing and one-to-one. Also, f is strictly positive, as we noted earlier when we said that the range of a^x is $(0, \infty)$. Notice too that $f(0) = 1$ and $f(1) = 2$.

Now consider $g(x) = \left(\frac{1}{2}\right)^x$. This looks like a completely new function, but $g(x) = f(-x)$ since $\left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$. Therefore, the graph of g is just the reflection of the graph of f across the y-axis!



The observations above lead us to the following theorem.

Theorem 8.1.3. *Let $a > 0$ with $a \neq 1$.*

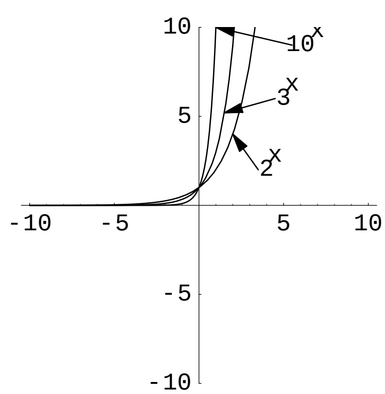
1. *If $a > 1$, the graph of $f(x) = a^x$ will rise to the right.*

2. *If $a < 1$, then the graph of $f(x) = a^x$ will fall to the right.*

3. *The graph of $g(x) = \left(\frac{1}{a}\right)^x$ is the reflection of the graph of $y = a^x$ across the y-axis.*

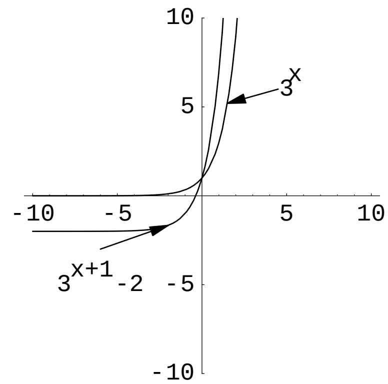
4. *If $f(x) = a^x$, then $f(0) = 1$ and $f(1) = a$.*

Naturally, a larger value of a will cause the graph to rise more rapidly. For purposes of comparison, we show below the graphs of $f(x) = 2^x, 3^x$, and 10^x.



We may also transform these graphs according to the same principles we use to transform other graphs.

Example 8.1.4. $f(x) = 3^{x+1} - 2$ has the same graph as $y = 3^x$, but is shifted one unit left and two units down.



Exponential functions are important because of their wide application in both business and the sciences.

Example 8.1.5. **Interest**. Suppose you invest P dollars at an annual interest rate of r (expressed as a decimal), and interest is compounded n times per year. Let $A(t)$ denote your account balance after t years have elapsed. Find a closed formula for $A(t)$ in terms of P, r, n, and t.

Solution. First, since r is an annual rate, the rate for a single compounding period is $\frac{r}{n}$. (For example, if interest is compounded monthly, then your rate for a month is $\frac{r}{12}$.)

Initially, your account holds P dollars. After one compounding period, you add to this interest in the amount of $\frac{r}{n}(P)$, so that your balance is

$$A\left(\frac{1}{n}\right) = P + \frac{r}{n}(P) = P\left(1 + \frac{r}{n}\right).$$

This is your beginning balance for the next period. (We use $t = \frac{1}{n}$ since one period out of n has elapsed.)

The amount of money at the beginning of the period is irrelevant to the above computation: if you begin the period with x dollars, you end the period with $\left(1 + \frac{r}{n}\right) x$ dollars. Therefore, since you begin the second period with $P\left(1 + \frac{r}{n}\right)$ dollars, you end it with

$$A\left(\frac{2}{n}\right) = P\left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$$

dollars.

This is what you begin the third compounding period with, so you end the third compounding period with

$$A\left(\frac{3}{n}\right) = P\left(1 + \frac{r}{n}\right)^2\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^3$$

dollars.

Notice that in each case, the exponent on $\left(1 + \frac{r}{n}\right)$ is n times the argument of A. For example, the exponent on $\left(1 + \frac{r}{n}\right)$ in $A(\frac{3}{n})$ is 3. In general, then, we have

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Example 8.1.6. Suppose \$1000 is invested at 8.2% for 5 years. Find the account balance if it is compounded annually, quarterly, monthly, and daily.

Solution: Recall that $A(t) = P\left(1+\dfrac{r}{n}\right)^{nt}$, where P is the amount invested, r is the interest rate (as a decimal), t is time elapsed in years, and n is the number of compounding periods per year. Thus, we have $P = 1000, r = 0.082$ and $t = 5$ with n varying in each case.

Annual compounding means once per year, so we have $n = 1$, and we get $A(5) = 1000(1+0.082)^5 = 1000(1.082)^5 \approx 1482.98$ dollars.

Quarterly compounding means four times per year, and we get $A(5) = 1000\left(1+\dfrac{0.082}{4}\right)^{4(5)} \approx 1500.58$ dollars.

Monthly compounding means 12 times per year, and we get $A(5) = 1000\left(1+\dfrac{0.082}{12}\right)^{12(5)} \approx 1504.72$ dollars.

Finally, daily compounding means 365 times per year, so we get $A(5) = 1000\left(1+\dfrac{0.082}{365}\right)^{365(5)} \approx 1506.75$ dollars.

It is reasonable that the interest grows more rapidly when the compounding is more frequent. What is perhaps surprising is that compounding annually versus compounding daily made less than \$25 difference over the course of five years. Just to satisfy our curiosity, let's see how much difference it makes if we let interest accrue for 25 years.

Compounding annually: $A(25) = 1000(1.082)^{25} \approx 7172.68$ dollars.

Compounding daily: $A(25) = 1000\left(1+\dfrac{0.082}{365}\right)^{365(25)} \approx 7766.11$ dollars.

Now the difference is nearly \$600.

Example 8.1.7. **Radioactive decay**. Let $A(t)$ be the amount in grams of a radioactive substance. Let A_0 be the initial amount (the amount at $t = 0$), so that $A(0) = A_0$. Let k be the half-life (the time it takes for half of the substance to decay). After one half-life, we have $A(k) = \frac{1}{2}A_0$. After two half-lives, we have $A(2k) = \frac{1}{2}\left(\frac{1}{2}A_0\right) = \frac{1}{4}A_0$. Notice that it doesn't matter how much you start with, after a half-life elapses, you have exactly half that much left. After t years have gone by, $\frac{t}{k}$ half-lives have gone by. For example, if the half-life is 4 years, then after three years, $\frac{3}{4}$ of a half-life has elapsed. Thus we have

$$A(t) = A_0\left(\frac{1}{2}\right)^{t/k}.$$

Example 8.1.8. Suppose that a certain material has a half life of 25 years, and there are

$$A(t) = 10(\frac{1}{2})^{t/25}$$

grams remaining after t years. Find the initial amount and the amount after 80 years.

Solution: The initial amount is 10g, as we can read directly off of the formula for $A(t)$. Alternatively, "initial amount" means the amount when no time has gone by, at $t = 0$, so we can simply compute $A(0) = 10(1/2)^0 = 10$ grams. Thus, after 80 years, we have $A(80) = 10\left(\frac{1}{2}\right)^{80/25} \approx 1.088$ grams left.

Exercises

Use your calculator to approximate each of the following to the nearest ten-thousandth, when possible.

1. $(2.3)^5$
2. $(1.4)^{-2}$
3. $(3.18)^{2.35}$
4. $(\sqrt{2})^{-3}$
5. $(\sqrt{\pi})^{-4}$
6. $(-3)^{\sqrt{2}}$
7. $-3^{\sqrt{2}}$
8. 100^{π}
9. $-100^{3\pi}$
10. $(5000)^{\frac{3}{25}}$
11. $\left(\frac{2}{3}\right)^{-\sqrt{2}}$
12. $\left(\frac{\sqrt{2}}{5}\right)^{7}$
13. $\left(\frac{1}{9}\right)^{-10}$

Sketch the graph of each exponential function, and explicitly evaluate each function at least four values of x.

14. $f(x) = \left(\frac{1}{3}\right)^x$
15. $f(x) = 4^{-x}$
16. $f(x) = 5^x$
17. $f(x) = \left(\frac{2}{3}\right)^{-x}$
18. $f(x) = \pi^x$
19. $f(x) = \left(\frac{5}{4}\right)^{-x}$
20. $f(x) = 3^{x-1} + 3$
21. $f(x) = 2^{x+4} - 2$

22. Use the graphs of $f(x) = 3^x$ and $g(x) = 4^x$ to solve the inequality $3^x < 4^x$.
23. Use the graphs $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = \left(\frac{1}{4}\right)^x$ to solve the inequality $\left(\frac{1}{4}\right)^x \leq \left(\frac{1}{3}\right)^x$.
24. Compute the value of an account of \$12,000 after four years if the interest rate is 7% and is compounded
    (a) monthly.
    (b) quarterly.
    (c) semiannually.
    (d) weekly.
    (e) daily.
    (f) hourly.
25. Compute the *interest* earned on a CD of \$1500 after 18 months if the interest rate is 8% and is compounded monthly.
26. Compute the *interest* earned on an investment of \$5000 after 18 months if the interest rate is 8.25% and is compounded daily.

27. If you had \$5,000 to invest for 4 years with the goal of greatest return on your investment, would you rather invest in an account paying

    (a) 7.2% compounded quarterly,

    (b) 7.15% compounded daily, or

    (c) 7.1% compounded hourly?

28. Many credit cards have interest rates of around 19%. Compute the interest on a balance of \$1000 after one year if the interest is compounded daily. (Assume that you do not make any payments; if you paid enough in the previous month, some cards will allow you to skip payments.)

29. A certain radioactive substance has a half-life of 1200 years. If a sample of the substance is 1000 grams, how much of the substance will be left in 10,000 years?

30. The half-life of uranium-235 is approximately 710,000,000 years. Of a 10-gram sample, how much will remain after 2000 years?

31. Estimate each quantity to the nearest hundred-thousandths.

    (a) $\left(1+\frac{1}{1}\right)^{1}$

    (b) $\left(1+\frac{1}{10}\right)^{10}$

    (c) $\left(1+\frac{1}{100}\right)^{100}$

    (d) $\left(1+\frac{1}{1000000}\right)^{1000000}$

    (e) Make a conjecture about the value of $\left(1+\frac{1}{x}\right)^{x}$ as x grows very large.

8.2 Logarithmic Functions

In the previous section we observed that exponential functions are one-to-one; this implies that they have inverse functions under composition.

Definition 8.2.1. The **logarithmic function** g with **base** a is the inverse of the function $f(x) = a^x$ for $a > 0, a \neq 1$. We write $g(x) = \log_a(x)$. That is,

$$y = \log_a(x) \text{ if and only if } a^y = x.$$

The domain of $\log_a$ is $(0, \infty)$, and the range is $(-\infty, \infty)$.

One way to think about logarithms is to note that the number $\log_a(x)$ answers the question, "To what power must one raise a to get x?"

Example 8.2.2. Using the definition above, we see that

1. $\log_2(8) = 3$, since $2^3 = 8$;
2. $\log_3(81) = 4$, since $3^4 = 81$;
3. $\log_{16}(4) = \frac{1}{2}$, since $16^{\frac{1}{2}} = 4$; and
4. $\log_{10}\left(\frac{1}{100}\right) = -2$, since $10^{-2} = \frac{1}{100}$.

Note that the domain of the function $\log_a(x)$ is the range of the function a^x, and the range of $\log_a(x)$ is the domain of a^x. We will sometimes write $\log_a x$ for $\log_a(x)$. Also, the symbol $\log_a$ represents a *function*, while $\log_a(x)$ represents a *number*; they are two entirely different kinds of objects. This is a somewhat subtle difference that may seem unimportant; however, if you take the time to understand it, it will help you avoid some serious computational errors. (See the unit on functions for more detail.)

Finally, since the functions a^x and $\log_a(x)$ are inverse functions, if we compose them, we get the identity function. Thus,

$$\log_a(a^x) = x \text{ and } a^{\log_a(x)} = x.$$

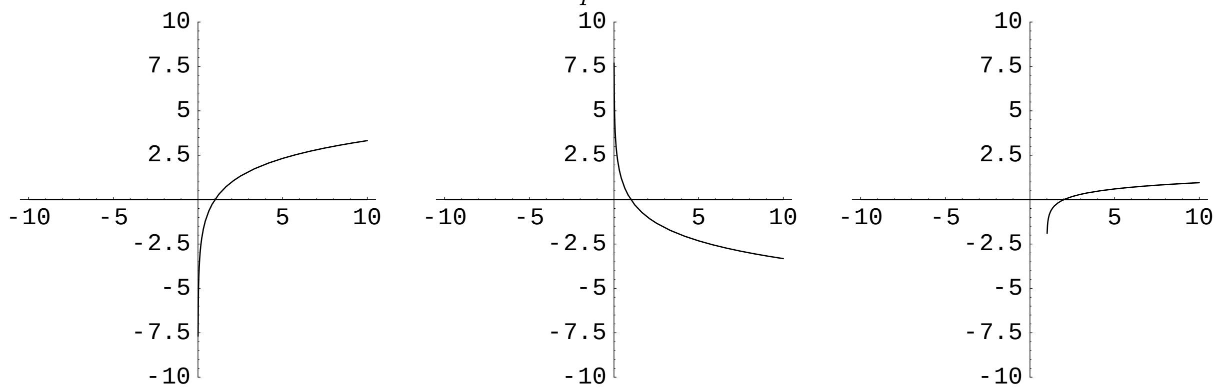
This is just the definition of inverse functions, but these identities are extremely important and will be essential when we begin solving equations.

Example 8.2.3. We have that

1. $\log_2(2^x) = x$,
2. $\log_a(a^3) = 3$,
3. $4^{\log_4(x)} = x$, while
4. $2^{\log_2(-5)} \neq -5$ since -5 is not in the domain of $g(x) = \log_2(x)$.

The graph of $\log_a(x)$ is not too hard to find since we already have graphs for exponential functions a^x; we need only reflect the graph of a^x across the line $y = x$. We may, as always, manipulate the graphs according to principles we have discussed. Thus, below, the graph of $\log_{10}(x-1)$ is just the graph of $\log_{10}(x)$ shifted to the right 1 unit.

Example 8.2.4.



$\log_2(x)$ $\quad$ $\log_{\frac{1}{2}}(x)$ $\quad$ $\log_{10}(x-1)$

Example 8.2.5. Find the domain of $f(x) = \log_4(3x+5)$.

Solution: The domain of $\log_a(x)$ is $(0, \infty)$ regardless of a, so we must have $3x + 5 > 0$. Thus $x > -\frac{5}{3}$, and the domain of f is $(-\frac{5}{3}, \infty)$.

The logarithm has properties corresponding to the properties of exponential functions. This is pretty reasonable since we defined logarithms based on exponentials. We summarize the properties of logarithms in the following theorem.

Theorem 8.2.6. *Let* $a > 0$, $a \neq 1$, *let* $b, x, y \in \mathbb{R}$, *and let* $u, v > 0$.

	Laws of Exponents	*Laws of Logarithms*
1	$a^0 = 1$	$\log_a(1) = 0$
2	$a^1 = a$	$\log_a(a) = 1$
3	$a^{\log_a x} = x$	$\log_a(a^x) = x.$
4	$a^x a^y = a^{x+y}$	$\log_a(uv) = \log_a u + \log_a v$
5	$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a \dfrac{u}{v} = \log_a u - \log_a v$
6	$(a^x)^y = a^{xy}$	$\log_a(u^b) = b \log_a u.$

Proof. We are already familiar with the properties of exponential functions; our task is to prove the properties of logarithmic functions.

1. To what power must we raise a to get 1? We raise a to the zero power: $a^0 = 1$. Thus $\log_a(1) = 0$.

2. To what power must we raise a to get a? We raise a to the first power: $a^1 = a$. Thus $\log_a(a) = 1$.

3. To what power must we raise a to get a^x? We raise a to the x power! Thus $\log_a(a^x) = x$.

4. Let $x = \log_a u$ and $y = \log_a v$. Then $a^x = u$ and $a^y = v$, so

$$\begin{aligned}\log_a(uv) &= \log_a(a^x a^y)\\ &= \log_a(a^{x+y})\\ &= x + y\\ &= \log_a u + \log_a v.\end{aligned}$$

   This is a template for the way these proofs go: we must rewrite our logarithm statements in terms of exponentials.

5. We leave the proof of this as an exercise.

6. Let $x = \log_a u$, so that $u = a^x$. Then

$$\begin{aligned}\log_a(u^b) &= \log_a[(a^x)^b]\\ &= \log_a(a^{bx})\\ &= bx\\ &= b\log_a u\end{aligned}$$

□

Notice that $\log_a(u^b)$ and $(\log_a u)^b$ are two different things. The placement of the parentheses makes a big difference here; the theorem does not apply to $(\log_a u)^b$.

Example 8.2.7. Suppose you are told that $\log_a(2) \approx 0.3562$ and $\log_a(3) \approx 0.5646$. Find

1. $\log_a(6)$,
2. $\log_a(1.5)$, and
3. $\log_a(9)$.

Solution:

1. $\log_a(6) = \log_a(2 \cdot 3) = \log_a(2) + \log_a(3) \approx 0.3562 + 0.5646 = 0.9208$.
2. $\log_a(1.5) = \log_a\left(\frac{3}{2}\right) = \log_a 3 - \log_a 2 \approx 0.5646 - 0.3562 = 0.2084$.
3. $\log_a(9) = \log_a(3^2) = 2\log_a(3) \approx 2(.5646) = 1.1292$.

Example 8.2.8. Expand $\log_{10} \dfrac{5}{x^3 y}$.

Solution:

$$\begin{aligned}\log_{10}\frac{5}{x^3y} &= \log_{10}(5) - \log_{10}(x^3y)\\ &= \log_{10}(5) - (\log_{10}(x^3) + \log_{10}(y))\\ &= \log_{10}5 - 3\log_{10}x - \log_{10}y.\end{aligned}$$

Example 8.2.9. Express $2\log_3(x+3) + \log_3(x) - \log_3 7$ as a single logarithm.

Solution:

$$\begin{aligned} 2\log_3(x+3) + \log_3(x) - \log_3 7 &= \log_3(x+3)^2 + \log_3\left(\frac{x}{7}\right) \\ &= \log_3\left(\frac{x(x+3)^2}{7}\right). \end{aligned}$$

Example 8.2.10. Condense the logarithmic expression $\frac{1}{2}\log_{10}(x) + 3\log_{10}(x+1)$.

Solution:

$$\begin{aligned} \frac{1}{2}\log_{10}(x) + 3\log_{10}(x+1) &= \log_{10}(x^{\frac{1}{2}}) + \log_{10}(x+1)^3 \\ &= \log_{10}(\sqrt{x}\cdot(x+1)^3). \end{aligned}$$

It is perhaps worth noting that there are no formulas for rewriting a logarithm of a sum or difference.

Exercises

Compute each logarithm.

1. $\log_8(512)$
2. $\log_{\frac{1}{3}}(27)$
3. $\log_5(625)$
4. $\log_2(16^5)$
5. $\log_6\left(\frac{1}{36}\right)$
6. $\log_{10}(0.00001)$

Sketch the graph of each logarithmic function.

7. $f(x) = \log_4(x)$
8. $f(x) = -\log_2(x)$
9. $f(x) = \log_5(x+2)$
10. $f(x) = -\log_2(x-3)$
11. $f(x) = \log_3(x-1) + 3$
12. $f(x) = \log_{2/3}(x) + 4$

Expand each logarithm.

13. $\log_4(3x^4)$
14. $\log_5(2x)$
15. $\log_6\left(\frac{x}{3}\right)$
16. $\log_3(5xy)$
17. $\log_5(x(x+1)^3)$
18. $\log_{12}((x+2)^3(x^2+1)^4)$
19. $\log_2\left(\frac{1}{2}\right)$
20. $\log_7\left(\frac{xy}{2z}\right)$
21. $\log_3\left(\frac{2x^2(x-4)^3}{(x+1)^4}\right)$
22. $\log_2\left(\frac{1}{x}\right)$

Condense each expression into a single logarithm.

23. $\log_3(x) + \log_3(2)$
24. $\log_5(z) - \log_5(y)$
25. $3\log_2(x+y)$
26. $-4\log_3(2x)$
27. $\frac{3}{2}\log_7(x-5)$
28. $-\frac{2}{3}\log_7(x+5)$
29. $\log_2(3x) - \log_2(x+3) + \log_2(x)$
30. $\log_5(4x) + 3\log_5(x-1)$
31. $2\log_{10}(3x+2y) - 2\log_{10}(6x+4y)$
32. $\log_6(x^2) - \log_6(2x) + \log_6\left(\frac{2}{x}\right)$

If $\log_a(10) \approx 2.3026$ and $\log_a(8) \approx 2.0794$, estimate the following:

33. $\log_a(80)$
34. $\log_a\left(\frac{4}{5}\right)$
35. $\log_a(640)$
36. $\log_a(1.25)$
37. $\log_a(512)$
38. $\log_a(6.4)$

39. Use the graphs $f(x) = \log_3 x$ and $g(x) = \log_4 x$ to solve the inequality $\log_3 x < \log_4 x$.
40. Use the graphs $f(x) = \log_{\frac{1}{3}}(x)$ and $g(x) = \log_{\frac{1}{4}}(x)$ to solve the inequality $\log_{\frac{1}{4}}(x) \leq \log_{\frac{1}{3}}(x)$.
41. Prove part 5 of Theorem 8.2.6.

8.3 The Natural Exponential and Logarithm Functions

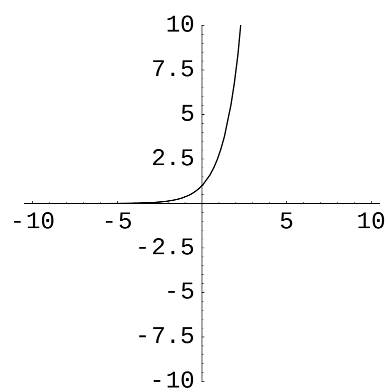
So far, we have been using any positive number (except 1) as a base for an exponential or logarithmic function. Now we want to focus in on one particular base. This base is very special; we choose it for properties that one needs to take calculus to fully appreciate, but one can still see its utility without going so far.

Definition 8.3.1. The irrational number e is approximately $2.71828\ldots$. It is the base of the **natural exponential function** and the **natural logarithm function**.

When you encounter e in your reading or in exercises, just remember that it is a symbol we use to indicate a specific, very special number, in the same way we use π to represent a specific, very special number.

Definition 8.3.2. The function $f(x) = e^x$ is the **natural exponential function**.

You will need to use your calculator to graph it or evaluate it. One interesting fact is that as x grows infinitely large, $\left(1 + \frac{1}{x}\right)^x$ comes arbitrarily close to e. This is explored more fully in an exercise.



$f(x) = e^x$

Example 8.3.3. Approximate each of the following.

1. $e^3 \approx 20.08553692$

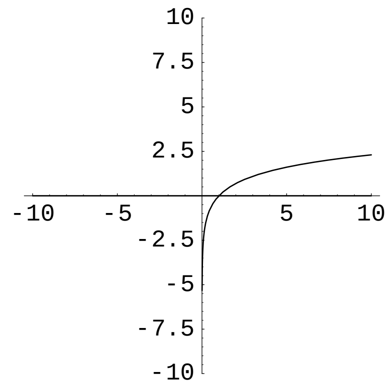
2. $e^{\frac{1}{2}} = \sqrt{e} \approx 1.648721271$

3. $\frac{3}{2}\left(e^{\pi}\right) \approx 34.71103895$

4. $e^{-2} \approx .1353352832$

Since the function $f(x) = e^x$ is one-to-one, we also need the inverse to e^x.

Definition 8.3.4. The **natural logarithm function** is defined by $f(x) = \log_e x$ for $x > 0$. It is usually written $\ln x$. Its graph is given below.



$f(x) = \ln x$

Of course, the natural logarithm obeys the same laws as the other logarithm functions. We have merely singled out a special base.

Example 8.3.5. Evaluate each of the following.

1. $\ln\left(\frac{1}{e}\right) = \log_e\left(\frac{1}{e}\right) = -1$
2. $\ln(e^2) = \log_e(e^2) = 2$
3. $\ln 6 \approx 1.7918$. (This required an approximation on a calculator.)
4. $\ln(x+2)^3 = 3\ln(x+2)$
5. $3\ln(x) + 5\ln(y) = \ln(x^3) + \ln(y^5) = \ln(x^3 \cdot y^5)$

Definition 8.3.6. The **common logarithm function** is given by $f(x) = \log_{10} x$ for $x > 0$. It is usually written simply as $\log x$.

Example 8.3.7.

1. $\log 10 = 1$.
2. $\log 100 = 2$.
3. $\log\left(\frac{1}{1000}\right) = -3$
4. $\log 12 \approx 1.0792$. (This required an approximation on a calculator.)

Your calculator is probably only set up to do logarithms to the bases 10 and e. What do we do if we need to compute a logarithm to another base? Somehow, we must express our logarithms in either base e or base 10. The theorem below tells us how to do this.

Theorem 8.3.8 (Change of Base Formula). *Let a, b, x be positive real numbers with $a, b \neq 1$. Then* $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Proof. Let $y = \log_a x$, so that $a^y = x$. Then

$$\begin{aligned}\log_b x &= \log_b a^y \\ &= y \log_b a \\ &= (\log_a x)(\log_b a).\end{aligned}$$

Therefore, since $\log_b x = (\log_a x)(\log_b a)$, we have $\log_a x = \dfrac{\log_b x}{\log_b a}$. $\square$

Example 8.3.9. Approximate $\log_5 12$.

Solution: We have $a = 5$ and $x = 12$. We will use $b = 10$; that is, we will convert this to a base 10 logarithm: $\log_5 12 = \dfrac{\log 12}{\log 5} \approx 1.5440$.

We could have chosen to convert this to base e, in which case we would have found $\log_5 12 = \dfrac{\ln 12}{\ln 5} \approx 1.5440$, as before.

Note that the natural logarithm is not the same as the common logarithm, but either may be used in the change of base formula given above.

Exercises

Evaluate each logarithm. (Use your calculator as necessary to approximate the values.)

1. $\log 1000$
2. $\ln e^{-3}$
3. $\ln 13$
4. $\log 103$
5. $\log_8 14$
6. $\log_{13} 169$
7. $\log_{1/3} 42$
8. $\ln(\ln 31)$

9. Sketch the graph of e^{3x}.
10. Sketch the graph of $f(x) = 2\ln(x + 3) - 1$.

Expand each logarithm.

11. $\log(2x)$
12. $\ln\left(\frac{x}{3}\right)$
13. $\log(5x^2y)$
14. $\ln(12x^4)$
15. $\log(x(x + 1)^3)$
16. $\ln((x + 5)^2(x^2 + 1)^3)$
17. $\ln\left(\frac{1}{e}\right)$
18. $\ln\left(\frac{e^x}{2x}\right)$
19. $\log\left(\frac{2x^2(x-4)^3}{(x+1)^4}\right)$
20. $\ln\left(\frac{1}{x}\right)$

Condense each expression into a single logarithm.

21. $\log(3x) + \log(2)$
22. $\ln(z) - \ln(y)$
23. $3\log(x + y)$
24. $-4\ln(2x)$
25. $\frac{3}{2}\ln(x - 5)$
26. $-\frac{2}{3}\log(x + 5)$
27. $\log(3x) - \log(x + 2) + \log(x)$
28. $\ln(4x) + 3\ln(x - 1)$
29. $2\log(3x + 2y) - 2\log(6x + 4y)$
30. $\ln(x^3) - \ln(3x) + \log\left(\frac{3}{x}\right)$

31. Compute each quantity.

    (a) $\left(1 + \dfrac{1}{1}\right)^1$

    (b) $\left(1 + \dfrac{1}{10}\right)^{10}$

    (c) $\left(1 + \dfrac{1}{100}\right)^{100}$

    (d) $\left(1 + \dfrac{1}{1000000}\right)^{1000000}$

    (e) $\left(1 + \dfrac{1}{10^{14}}\right)^{10^{14}}$ Can you explain what went wrong?

    (f) Graph $f(x) = \left(1 + \dfrac{1}{x}\right)^x$ on your calculator. With your window set to $[0, 1000000000000]$, use the trace feature to determine the y-values on the graph. What do you notice?

8.4 Exponential and Logarithmic Equations

We want, as usual, to be able to solve equations involving our new functions. We will need to use the properties of exponents and logarithms to do this. It is important to note that if f is a function and $x = y$, then $f(x) = f(y)$. In particular, we need to know that if $x = y$, then $a^x = a^y$ and $\log_a x = \log_a y$. This allows us to solve equations, as you will see in the following example.

Example 8.4.1. Solve each equation.

1. $\log_3(4x - 7) = 2$
2. $2\log_5 x = \log_5 9$
3. $3^{x+1} = 81$
4. $4^x - 2^x - 12 = 0$
5. $5^{x-2} = 3^{3x+2}$

Solution:

1.

$$\begin{array}{rcll} \log_3(4x-7) & = & 2 & \text{Given} \\ 3^{\log_3(4x-7)} & = & 3^2 & \text{Apply } f(x) = 3^x \text{ to both sides} \\ 4x - 7 & = & 9 & \text{Inverse Functions} \\ 4x & = & 16 & \text{Add 7 to both sides} \\ x & = & 4 & \text{Divide by 4 on both sides} \end{array}$$

Check that $x = 4$ is a solution.

2.

$$\begin{array}{rcll} 2\log_5 x & = & \log_5 9 & \text{Given} \\ \log_5 x^2 & = & \log_5 9 & \text{Property of Logarithms} \\ 5^{\log_5 x^2} & = & 5^{\log_5 9} & \text{Apply } f(x) = 5^x \text{ to both sides} \\ x^2 & = & 9 & \text{Inverse Functions} \\ x & = & \pm 3 & \underline{\hspace{6cm}} \end{array}$$

However, a solution must solve the *original* equation, and $x = -3$ does not. Remember that the domain of any logarithm is $(0, \infty)$, so $\log_a(-3)$ does not make sense for any a. You should check that $x = 3$ is a solution.

3.

$$\begin{array}{rcll} \log_3(3^{x+1}) & = & \log_3(81) & \text{Given} \\ x + 1 & = & 4 & a^{\log_a x} = x \\ x & = & 3 & \text{Subtract 1 from both sides} \end{array}$$

Check that $x = 3$ is a solution.

4. This one looks very difficult at first, but if we rewrite it cleverly as $(2^x)^2 - 2^x - 12 = 0$, then we can recognize it as a disguised quadratic.

$$\begin{array}{rcll}
4^x - 2^x - 12 & = & 0 & \text{Given} \\
(2^x)^2 - 2^x - 12 & = & 0 & 4^x = (2^2)^x = 2^{2x} = 2^{x2} = (2^x)^2 \\
z^2 - z - 12 & = & 0 & \text{Substitute } z = 2^x \\
(z-4)(z+3) & = & 0 & \text{Factor} \\
z - 4 = 0 & \text{or} & z + 3 = 0 & \text{Zero Product Theorem} \\
z = 4 & \text{or} & z = -3 & \text{Add equals to equals} \\
2^x = 4 & \text{or} & 2^x = -3 & \text{Substitute } z = 2^x \\
x = 2 & & & \text{Inspection}
\end{array}$$

Note that since the range of a^x is $(0, \infty)$ for any a, $2^x = -3$ is not possible. Check that $x = 2$ is a solution.

5. Up until now, we chose the base for the logarithm to use by seeing what the base on the exponentials was; this time, however, we have two different bases! Should we apply $\log_5$ to both sides, or $\log_3$? Actually, it doesn't matter what base we decide to use. (Why?) For convenience, we will take the natural logarithm of both sides.

$$\begin{array}{rcll}
5^{x-2} & = & 3^{3x+2} & \text{Given} \\
\ln(5^{x-2}) & = & \ln(3^{3x+2}) & \text{Apply } f(x) = \ln x \text{ to both sides} \\
(x-2)\ln 5 & = & (3x+2)\ln 3 & \text{Property of Logarithms} \\
x\ln 5 - 2\ln 5 & = & 3x\ln 3 + 2\ln 3 & \text{Distributive Law} \\
x\ln 5 - 3x\ln 3 & = & 2\ln 5 + 2\ln 3 & \text{Add equals to both sides} \\
x(\ln 5 - 3\ln 3) & = & \ln 5^2 + \ln 3^2 & \text{Distributive Law} \\
x & = & \dfrac{\ln 5^2 + \ln 3^2}{\ln 5 - 3\ln 3} & \text{Divide on both sides} \\
x & = & \dfrac{\ln(25 \cdot 9)}{\ln 5 - \ln 27} & \text{Properties of Logarithms} \\
x & = & \dfrac{\ln 225}{\ln\left(\frac{5}{27}\right)} & \text{Properties of Logarithms} \\
x & \approx & -3.2116 & \text{Calculator approximation}
\end{array}$$

We summarize the above methods below.
To solve an **exponential equation:**

1. Isolate the exponential expression.
2. Take the appropriate logarithm of both sides.
3. Solve for the variable.

We outline an analogous procedure to solve logarithmic equations.

To solve a **logarithmic equation:**

1. Isolate the logarithmic expression (write each side as a single logarithm).

2. Exponentiate both sides. (That is, raise both sides to the power which is the base of the logarithms you're dealing with, or a convenient base if the problem involves more than one base.)

3. Solve for the variable.

Example 8.4.2. 1. Solve $\ln x + \ln 7 = 2$.

Solution:

$$\begin{array}{rcll} \ln x + \ln 7 & = & 2 & \text{Given} \\ \ln(7x) & = & 2 & \text{Property of Logarithms} \\ e^{\ln(7x)} & = & e^2 & \text{Exponentiate both sides} \\ 7x & = & e^2 & \text{Inverse Functions} \\ x & = & \frac{e^2}{7} & \text{Divide by 7} \\ x & \approx & 1.05579 & \text{Calculator approximation} \end{array}$$

Check that this is indeed a solution to the original equation.

2. $\log_2(x+5) - \log_2(x-2) = 3$

Solution:

$$\begin{array}{rcll} \log_2(x+5) - \log_2(x-2) & = & 3 & \text{Given} \\ \log_2 \dfrac{x+5}{x-2} & = & 3 & \text{Property of Logarithms} \\ \dfrac{x+5}{x-2} & = & 2^3 & \text{Apply } f(x) = 2^x \text{ to both sides, inverse functions} \\ \dfrac{x+5}{x-2} & = & 8 & 2^3 = 8 \\ x+5 & = & 8(x-2) & \text{Fraction Equivalence} \\ x+5 & = & 8x - 16 & \text{Gen. Distributive Law} \\ 21 & = & 7x & \text{Add/subtract equals} \\ x & = & 3 & \text{Divide by 7, symmetry of equality} \end{array}$$

Check this.

3. $\ln x + \ln(2-x) = 0$.

Solution:

$$
\begin{array}{rcll}
\ln x + \ln(2-x) & = & 0 & \text{Given} \\
\ln(x(2-x)) & = & 0 & \text{Property of Logarithms} \\
2x - x^2 & = & e^0 & \text{Exponentiate both sides} \\
2x - x^2 & = & 1 & e^0 = 1 \\
-x^2 + 2x - 1 & = & 0 & \text{Subtract 1, Summand Permutation} \\
x^2 - 2x + 1 & = & 0 & \text{Symmetry of equality} \\
(x-1)^2 & = & 0 & \text{Factor} \\
(x-1) & = & 0 & \text{Zero Product Theorem} \\
x & = & 1 & \text{Add 1 to both sides}
\end{array}
$$

Check that this is a solution.

Example 8.4.3. Carbon 14 is a radioactive isotope of carbon 12, with a half-life of about 5700 years. When an organism dies, its carbon 14 is at the same level as that of the environment, but then it begins to decay. We can determine an object's age by knowing the percentage of carbon-14 that remains. (This actually involves some big, not necessarily very good, assumptions that we will not go into here.) We know from section 8.1 that

$$A(t) = A_0 \left(\frac{1}{2}\right)^{t/5700},$$

where $A(t)$ is the amount of carbon 14 remaining after t years, and $A(0)$ is the amount of carbon 14 present at time 0 (when the organism died).

Example 8.4.4. Charcoal from an ancient tree that was burned during a volcanic eruption has only 45% of the standard amount of carbon 14. When did the volcano erupt?

We know that $A(t) = A_0 \left(\frac{1}{2}\right)^{t/5700}$. We also know that $A(t) = 0.45A_0$, although we don't know what t is. (Note that $0.45A_0$ is 45% of A_0.) Thus

$$
\begin{array}{rcll}
A(t) & = & A_0 \left(\frac{1}{2}\right)^{t/5700} & \text{Given} \\
A(t) & = & 0.45A_0 & \text{Given} \\
A_0 \left(\frac{1}{2}\right)^{t/5700} & = & 0.45A_0 & \text{Substitution} \\
\left(\frac{1}{2}\right)^{t/5700} & = & 0.45 & \text{Divide by } A_0 \neq 0 \\
\ln\left(\left(\frac{1}{2}\right)^{t/5700}\right) & = & \ln(0.45) & \text{Take ln of both sides} \\
\frac{t}{5700} \ln\left(\frac{1}{2}\right) & = & \ln 0.45 & \text{Property of Logarithms} \\
t & = & 5700 \left(\frac{\ln 0.45}{\ln 0.50}\right) & \text{Multiply by equals} \\
t & \approx & 6566 & \text{Calculator approximation}
\end{array}
$$

Thus, the eruption occurred approximately 6566 years ago.
This technique is known as **carbon dating**.

Exercises

Solve each exponential equation. After finding an exact solution, give an approximation to the nearest thousandths.

1. $3^{x+2} = 27$
2. $2^{3x+1} = 17$
3. $5^{-x/2} = 125$
4. $1 + e^x = 13$
5. $e^{3x+2} = 5$
6. $3e^{x+2} = 75$
7. $\left(1 + \frac{.08}{12}\right)^{12x} = 2$
8. $\frac{1000}{1+e^x} = \frac{1}{2}$
9. $36^x - 3 \cdot 6^x = -2$
10. $4^{2x+1} = 5^{x-4}$

Solve each logarithmic equation. After finding an exact solution, give an approximation to the nearest thousandth.

11. $\ln x = 7.2$
12. $\log(x+2) = 75$
13. $\ln(x+2) = -2.6$
14. $\ln(x) - \ln(5) = 13$
15. $3.5\ln(x) = 8$
16. $\log_2(x) + \log_2(x+2) = 4$
17. $2\log(x) + \log(4) = 2$
18. $\log_4(x+1) - \log_4(x) = 1$
19. $\ln(x+1) + \ln(x-2) = \ln(x^2)$
20. $\log_2(x+1) - \log_2(x-2) = 3$

21. An amateur archeologist claims to have discovered a dinosaur fossil that is only 3000 years old. It contains very little carbon-14; the best you can tell, it has less than 0.01% of the normal amount. (Your instruments are very crude.) What do you tell the archeologist, and why?

22. A skin coat has only 39% of the normal amount of carbon-14. How long ago was the animal killed to make the coat?

23. How long will it take to produce $10,000 from a $7,000 investment at 8% compounded monthly?

24. What interest rate, compounded monthly, will produce $10,000 from a $7,000 investment in 5 years?