Chapter 9

Polynomial Functions

In this unit, we define and investigate polynomial functions as a special class of functions. We begin with a review of linear and quadratic functions before discussing polynomials of arbitrary degree. We present the "traditional" theory of equations in a concise format and tie the procedures to the basic algebraic properties previously presented.

Objectives

- To define polynomial functions
- To explore, in detail, the properties of linear and quadratic functions
- To investigate general properties of polynomial functions
- To increase knowledge about the graphs of polynomial functions
- To introduce techniques for solving polynomial equations

Terms

- polynomial function
- degree
- leading coefficient
- linear function
  - slope, parallel, perpendicular
- quadratic function
  - standard form, general form, completing the square
  - parabola, vertex
  - discriminant
- Leading Coefficient Test
- Division Algorithm (for Polynomial Functions)
- Remainder Theorem
- Factor Theorem
- Fundamental Theorem of Algebra
- Rational Root Test
- Synthetic Division

9.1 Linear Functions

Definition 9.1.1. A function f defined by $f(x) = mx + b$ for some constants m and b is called a **linear function**. The number m is called the **slope** of the line, and b is the y-**intercept**.

The number b is called the y-intercept because $f(0) = b$. This means that when $x = 0$ (that is, when the point is on the y-axis), $y = b$. Thus, the y-intercept is where the graph crosses the y-axis. (Some prefer to reserve the term y-intercept for the *point* $(0, b)$; we will not make this distinction.)

We have dealt with linear equations already. Just remember that you MUST use the algebraic principles we have developed. There is no magic to it!

Consider the following example:

Example 9.1.2. Peggy leaves for work in her car, and drives at 45 mph. Her husband Bob notices that Peggy has left her lunch behind. He hops in his car ten minutes later and chases her at 55 mph. How far will Bob go before he catches Peggy?

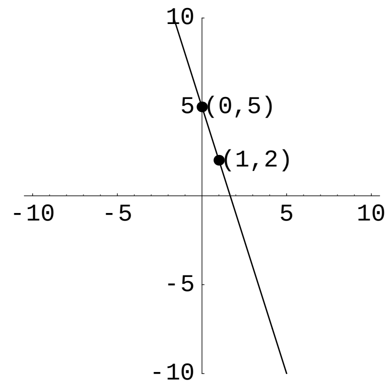
Solution: Let t be how long Peggy has driven. Then her distance is $45t$, and Bob's distance is $55(t-10)$. We need to know when these are equal. We solve: $45t = 55(t-10) \implies 45t = 55t - 550 \implies -10t = -550 \implies t = 55$. Therefore, Peggy has driven for 55 minutes and Bob has driven for 45 minutes. This means that Bob drives 55 mph for 3/4 of an hour, so he goes 41.25 miles.

There is nothing new here; the goal is for you to be able to think your way through setting up such problems. Solving them should then be a simple exercise in the algebraic principles we have studied.

We examined graphs in a previous section. Suppose that $f(x) = mx + b$. If $m = 0$, then $f(x) = b$, so we get a horizontal line. If $m \neq 0$, then $f(x) = m\left(x + \frac{b}{m}\right)$. We can now recognize this as a transformation of the line $y = x$. However, it is usually easier to obtain the graph of $f(x) = mx + b$ by plotting two points and connecting them with a line. A good choice for one point is the y-intercept.

Example 9.1.3. Sketch the graph of $f(x) = -3x + 5$.

Solution: The y-intercept is 5, so the point $(0, 5)$ is on the graph of f. Since $f(1) = 2$, we have that $(1, 2)$ is also on the graph of f. Since two points determine a line, this is sufficient for us to draw the rest of the graph.



Definition 9.1.4. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two distinct points with $x_1 \neq x_2$. The **slope** of the nonvertical line L containing P_1 and P_2 is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

If $x_1 = x_2$, then L is a vertical line and the slope m of L is undefined. (It is sometimes said that a vertical line has *no slope.* This is not to be confused with a horizontal line having *zero slope.*)

It is not too hard to show that this definition of slope is equivalent to the definition we gave previously.

Example 9.1.5. The slope of the line through the points $\left(1, -\dfrac{1}{3}\right)$ and $\left(\frac{4}{9}, -2\right)$ is

$$m = \frac{\left(-2 + \dfrac{1}{3}\right)}{\left(\dfrac{4}{9} - 1\right)} = \frac{\left(-\dfrac{5}{3}\right)}{\left(-\dfrac{5}{9}\right)} = 3.$$

If (x, y) is another point on this line, then $\dfrac{y - (-2)}{x - \left(\frac{4}{9}\right)} = 3$ as well. Thus, an equation of the line is $y + 2 = 3\left(x - \frac{4}{9}\right)$, or $y = 3x + \frac{2}{3}$.

Definition 9.1.6. Two distinct nonvertical lines L_1 and L_2 are **parallel** if and only if their slopes m_1 and m_2 are equal. (Any two distinct vertical lines are parallel.)

We are accustomed to thinking of parallel lines as being lines that do not intersect; we will show now that that definition corresponds to our definition above. If $y = mx + b_1$ and $y = mx + b_2$ are two distinct lines (so that $b_1 \neq b_2$), then they cannot intersect: if (x, y) is a point on both lines, then $y = mx + b_1$ (since (x, y) is on the first line), and $y = mx + b_2$ (since (x, y) is on the second line). Therefore, $mx + b_1 = mx + b_2$, so $b_1 = b_2$ by additive cancellation. But we already said that $b_1 \neq b_2$, so this is a contradiction. Since the only possible error was the assumption that the two lines intersect, they must not intersect, so they are parallel in the geometrical sense.

On the other hand, if two lines have different slopes, say, m_1 and m_2, then the point $\left(\dfrac{b_2 - b_1}{m_1 - m_2}, \dfrac{m_1 b_2}{m_1 - m_2}\right)$ lies on both lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$. (Check that.)

This means that lines intersect if and only if they have different slopes.

Example 9.1.7. Prove that the lines $2x - 3y + 7 = 0$ and $4x - 6y - 3 = 0$ are parallel.

Solution: Solving both equations for y we get $y = \frac{2}{3}x + \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{3}{6} = \frac{2}{3}x - \frac{1}{2}$. Since these lines both have slope $\frac{2}{3}$, they are parallel lines.

Definition 9.1.8. Two nonvertical lines L_1 and L_2 are **perpendicular** if and only if the product of their slopes m_1 and m_2 is -1. Alternatively, two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of each other. (A vertical line is perpendicular to a horizontal line.)

It can also be shown that this definition of perpendicular corresponds to the usual definition ("intersect at right angles") by means of the Pythagorean theorem, but that is more appropriate for a course in analytic geometry.

Example 9.1.9. Find an equation of the line perpendicular to $2x - 3y + 7 = 0$ and passing through the point $(1, 3)$.

Solution: Since $y = \frac{2}{3}x - \frac{7}{3}$, we know the slope of the given line is $m_1 = \frac{2}{3}$. The desired perpendicular line has slope $m_2 = -\frac{3}{2}$ and passes through the point $(1, 3)$. Thus, the desired line has equation $y - 3 = -\frac{3}{2}(x - 1)$. (This can be found from the fact that if (x, y) lies on the line, then $\frac{y-3}{x-1} = -\frac{3}{2}$.)

The following review of equations of lines may prove helpful.

Description	Equation
Vertical Line	$x = a$
Horizontal Line	$y = b$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_1 = m(x - x_1)$
Two-Point Form	$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
General Form	$Ax + By + C = 0$

Exercises

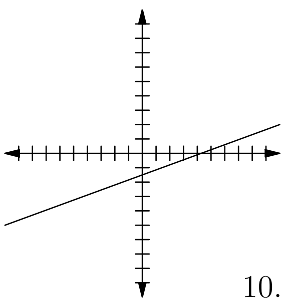
Determine whether the given pair of lines is parallel, perpendicular, or neither.

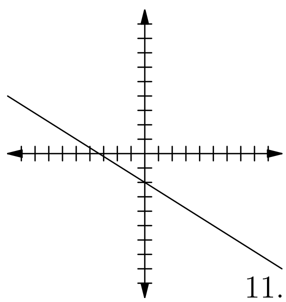
1. $y = 3x - 5$; $y = 3x + 2$
2. $6x - 4y + 8 = 0$; $9x - 6y + 2 = 0$
3. $2x + y = 4$; $4x - 8y = 0$
4. $4x + y - 2 = 6$; $y = 4x + 9$

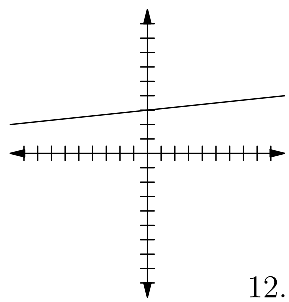
Graph the line through each pair of points.

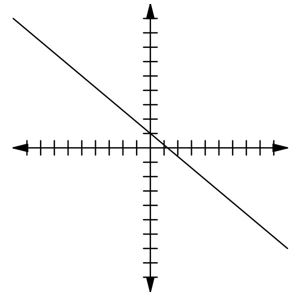
5. $(-1, 4)$ and $(2, 2)$
6. $(-2/3, 5)$ and $(-2/3, -2)$
7. $(4, -1)$ and $(-2, -1)$
8. $(2.4, 1.1)$ and $(0.5, 4.7)$

Find an equation of each line by first finding two points lying on the line.

9. 

10. 

11. 

12. 

13. Find an equation of the line that is perpendicular to the line $2x + 5y - 3 = 0$ and that passes through the point $(12, -7)$.

14. Find an equation of the line that is perpendicular to the line $y = 6x - 8$ and that passes through the point $(2, -4)$.

15. Find an equation of the line that is parallel to the line $2x + 5y - 3 = 0$ and that passes through the point $(12, -7)$.

16. Find an equation of the line that is parallel to the line $y = 6x - 8$ and that passes through the point $(2, -4)$.

17. JarCo, Inc. has observed that they will sell 2400 jars if the price of a jar is \$0.55, and that for every \$0.05 they raise the price, they sell 125 fewer jars. Find a linear function expressing the number of jars they sell as a function of the price they charge.

18. The distance from Houston to Dallas is approximately 250 miles. Marcie lives in Houston and her friend Bryson lives in Dallas; they leave their respective homes at the same time and drive toward the others' city.

    (a) If Bryson drives 65 miles per hour, what is his distance from Houston t hours after he has left Dallas?

(b) If Marcie drives 50 miles per hour, what is her distance from Houston t hours after she has left Houston?

(c) If Marcie and Bryson leave their homes at 8:00 in the morning, at what time will they be at the same place on the highway?

9.2 Quadratic Functions

Definition 9.2.1. A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$. A quadratic function f is in **standard form** if it is written as $f(x) = a(x-h)^2 + k$.

To put a function in standard form, we must **complete the square**. Recall from our earlier work (proving theorems in fields) that $(a+b)^2 = a^2 + 2ab + b^2$. Therefore, if we have $a^2 + 2ab + c$, we can "complete the square" by rewriting this number as

$$(a^2 + 2ab + b^2) - b^2 + c = (a+b)^2 - b^2 + c.$$

Since we both added and subtracted b^2, we really only added "a clever 0." Therefore, the final number is equal to the original number (because 0 is the additive identity). Notice that what we added (and subtracted) was the *square of half of the coefficient of* a. That is, the coefficient of a was $2b$, and we added (and subtracted) b^2.

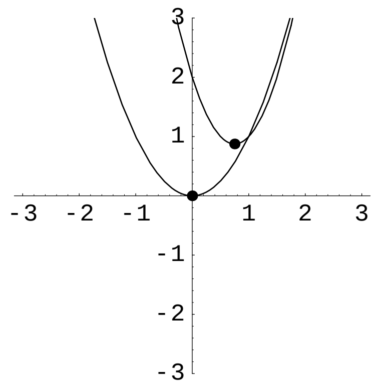
Example 9.2.2. Put $f(x) = x^2 - 6x + 2$ in standard form. We have $f(x) = x^2 + 2(-3)x + 2$. Thus our a is x, and our b is -3. We get

$$f(x) = (x^2 + 2(-3)x + (-3)^2) - (-3)^2 + 2 = (x + (-3))^2 - 9 + 2 = (x-3)^2 - 7.$$

Example 9.2.3. Put $f(x) = 2x^2 - 3x + 2$ in standard form. It is simpler to complete the square if the leading coefficient is 1. We rewrite f as

$$\begin{aligned} f(x) &= 2\left(x^2 - \frac{3}{2}x\right) + 2 \\ &= 2\left(x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2 - \left(-\frac{3}{4}\right)^2\right) + 2 \\ &= 2\left(x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2\right) - 2\cdot\left(-\frac{3}{4}\right)^2 + 2 \\ &= 2\left(x - \frac{3}{4}\right)^2 + \frac{7}{8}. \end{aligned}$$

These results also indicate certain features of the graph of a quadratic function, which is called a **parabola**, as noted in an earlier section. We can see in the example above that the graph of f is just a transformation of the graph of $y = x^2$. We have a translation to the right by $\frac{3}{4}$, followed by a vertical stretch by a factor of 2, and finally an upward shift by $\frac{7}{8}$. Thus, we can find the graph of f from prior knowledge.



Furthermore, we may ask what happens to the **vertex** of the parabola when we perform this transformation. (The vertex is the highest or lowest point of the parabola.)

The answer is that the vertex is shifted right by $\frac{3}{4}$ (as is every other point), the stretch does not affect the vertex (since the vertex is located at $y = 0$, and $2 \cdot 0 = 0$), and then the vertex is shifted up $\frac{7}{8}$ (as is every other point). The net result is that the vertex of f is located at $(\frac{3}{4}, \frac{7}{8})$.

We will generalize this idea in the following example.

Example 9.2.4. Consider a generic quadratic function $f(x) = ax^2 + bx + c$, with $a \neq 0$. We will complete the square in exactly the same way we did in Example 2.

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + 2\left(\frac{b}{2a}\right)x\right) + c \\
&= a\left(x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a\left(x - \frac{-b}{2a}\right)^2 - \frac{b^2}{4a} + c.
\end{aligned}$$

We can see here again a transformation of the graph of $y = x^2$. The graph is first shifted to the right by $\frac{-b}{2a}$. (Note that if $\frac{-b}{2a} < 0$, then this is really a shift to the left.) In particular, the vertex of the graph is translated to the point $\left(\frac{-b}{2a}, 0\right)$.

Next, the graph is stretched or compressed vertically by a factor of a. If it happens that $a < 0$, the graph is also reflected across the x-axis. Neither operation will affect the location of the vertex since it is on the x-axis.

Finally, the graph is shifted up by $-\frac{b^2}{4a} + c$. (This will be a downward shift if this quantity is negative.) This will move the vertex from $\left(\frac{-b}{2a}, 0\right)$ to $\left(\frac{-b}{2a}, -\frac{b^2}{4a} + c\right)$.

Thus, the vertex of the parabola $y = ax^2+bx+c$ has x-coordinate $\frac{-b}{2a}$. There is no point in memorizing the expression for the y-coordinate, as it can always be obtained by substituting the x coordinate into the function. (See the example below.) Furthermore, we saw that if $a < 0$, the parabola will be reflected across the x-axis, and therefore open downward. It is useful to note that henceforth we will not need to complete the square to determine this information: because we used general coefficients a, b, and c, we can apply the formula to any quadratic function as long as $a \neq 0$.

Parabolas are especially significant since projectiles travel in parabolic paths. Next time you see one of those huge sprinklers spraying crops, look carefully at the path of the water — it is a parabola (or nearly; our parabolic model does not account for wind resistance). Parabolas also possess many properties that make them useful in engineering; a course in analytic geometry will include such topics.

Example 9.2.5. A juggler in a parade juggles chainsaws on one of the floats. The height of a chainsaw above the ground x seconds after the juggler releases it is given by

$$h(x) = -16x^2 + 30x + 5$$

feet. Should the juggler rest while his float goes under a bridge with 15 feet of clearance?

Solution: We need to find the highest point reached by his chainsaw; that is, we need to find the vertex of the parabola the chainsaw describes as it is juggled. Notice that we have $a = -16$, so the parabola opens downward. This tells us that the parabola will have a maximum — a good thing if this problem is going to make sense!

The vertex of the parabola occurs at $x = \frac{-30}{2(-16)} = 0.9375$ seconds. This is not the maximum height, however, it is the **time** at which the maximum height is reached. To determine the height, evaluate $h(0.9375) = -16(0.9375)^2 + 30(0.9375) + 5 = 19.0625$ feet. The juggler should probably pause while the float goes under the bridge.

Sometimes we are not given an equation describing the function; we may only be given a few points that lie on its graph, and told to find the equation ourselves. This happens very frequently in the sciences. Scientists collect data in the course of an experiment, and they want to find some general rule which can be used to predict the outcomes of future experiments. Consider the following example.

Example 9.2.6. Find an equation of the quadratic function f whose graph passes through the points $(1, 3), (-1, 4)$, and $(2, 1)$.

Solution: If $(1, 3)$ is on the graph of f, that means (by definition) that $f(1) = 3$. Similarly, $f(-1) = 4$ and $f(2) = 1$. What does f itself look like? We are told that f is a quadratic function, so f must have the form $f(x) = ax^2 + bx + c$ for some a, b, and c which we need to determine.

Since $f(1) = 3$, we have

$$3 = f(1) = a(1)^2 + b(1) + c = a + b + c.$$

Since $f(-1) = 4$, we have

$$4 = f(-1) = a(-1)^2 + b(-1) + c = a - b + c$$

Finally, since $f(2) = 1$, we have

$$1 = f(2) = a(2)^2 + b(2) + c$$

Together, these give us the system of **linear** equations

$$\begin{aligned} a + b + c &= 3 \\ a - b + c &= 4 \\ 4a + 2b + c &= 1 \end{aligned}$$

We may write this as the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 \\ -1/2 \\ 4 \end{bmatrix}. \end{aligned}$$

That is, $a = -\frac{1}{2}, b = -\frac{1}{2}$, and $c = 4$. We finally arrive at $h(x) = -\dfrac{1}{2}x^2 - \dfrac{1}{2}x + 4$. You should check that all three of the given points lie on this parabola.

Exercises

Complete the square to write each quadratic function in standard form.

1. $f(x) = x^2 + 6x - 12$
2. $f(x) = x^2 + 5x + 3$
3. $f(x) = -2x^2 - 8x + 15$
4. $f(x) = 5x^2 - 3x + 2$

Find the vertex of each parabola.

5. $f(x) = x^2 + 8x + 15$
6. $f(x) = 3x^2 - 7x - 1$
7. $f(x) = -16x^2 + 48x + 80$
8. $f(x) = 2x^2 + 13$

Find an equation of a parabola through the given points.

9. $(-1, 2), (3, 5), (4, 2)$
10. $(-2, 3), (0, 0), (1, 4)$
11. $(-100, 20), (0, 50), (100, 20)$
12. $(0, 203), (0.25, 178), (0.45, 107)$. (These are three points from the dropped ball exercises in the function sections. Compare your parabola to the one given there.)
13. A punted football follows a parabolic path. Its height at $t = 0$ s is 3 feet, its height at $t = 2$ s is 95 feet, and its height at $t = 4$ s is 59 feet.
    (a) Find an equation of the path the football follows.
    (b) How high does the football go?
    (c) What is the hang-time of the football?

9.3 Solving Quadratic Equations

We found in the previous section that a **general** quadratic function $f(x) = ax^2 + bx + c$ may be put into the **standard** form

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

We wish now to find the zeros of our general quadratic; that is, we want to know for which values of x we have $f(x) = 0$. We have four basic algebraic techniques:

1. Factoring
2. Extracting Square Roots
3. Completing the Square
4. The Quadratic Formula

The method of factoring uses the Zero-Product Theorem: If $ab = 0$, then $a = 0$ or $b = 0$. That is, if we have a product (in a field) equal to zero, then we must conclude that at least one of the factors is zero.

Example 9.3.1. Solve the (quadratic) equation $x^2 - 3x - 10 = 0$ by factoring.

Solution:

$$\begin{array}{ccc} & x^2 - 3x - 10 = 0 & \\ & (x-5)(x+2) = 0 & \\ (x-5) = 0 & \text{or} & (x+2) = 0 \\ x = 5 & \text{or} & x = -2 \end{array}$$

Example 9.3.2. Solve the equation $9x^2 - 6x + 3 = 2$.

Solution:

$$\begin{aligned} 9x^2 - 6x + 3 &= 2 \\ 9x^2 - 6x + 1 &= 0 \\ (3x-1)(3x-1) &= 0 \\ (3x-1) &= 0 \\ x &= \frac{1}{3} \end{aligned}$$

The second method, extracting square roots, refers to "taking the square-root of both sides." Solving the equation $x^2 = a$ means finding all values of x that square to a. In this case, $x = \pm\sqrt{a}$, since $(\sqrt{a})^2 = a$ and $(-\sqrt{a})^2 = a$. Note that finding all x such that $x^2 = a$ is a different problem from finding $\sqrt{a}$ since our convention is that the symbol $\sqrt{\ }$ means the *principal* square root; therefore, when taking the square root of both sides of an equation, we must remember the $\pm$. (Why is putting $\pm$ on one side of the equation sufficient?) Note also that the method of extracting square roots is a special case of factoring, as shown in the examples below.

Example 9.3.3. Solve the equation $4x^2 = 12$.

Solution 1:

$$\begin{aligned} 4x^2 &= 12 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Solution 2:

$$\begin{gathered} 4x^2 = 12 \\ 4x^2 - 12 = 0 \\ 4(x^2 - 3) = 0 \\ x^2 - 3 = 0 \\ (x - \sqrt{3})(x + \sqrt{3}) = 0 \end{gathered}$$

$$\begin{array}{ccc} (x - \sqrt{3}) = 0 & \text{or} & (x + \sqrt{3}) = 0 \\ x = \sqrt{3} & \text{or} & x = -\sqrt{3} \end{array}$$

Example 9.3.4. Solve the equation $(x - 5)^2 = 15$.

Solution:

$$\begin{aligned} (x-5)^2 &= 15 \\ x - 5 &= \pm\sqrt{15} \\ x &= 5 \pm \sqrt{15} \end{aligned}$$

We see from the above example that if we have a perfect square involving the "variable side" of the equation, then we might extract square roots to solve the equation. In fact, we may force a perfect square involving the variable on one side of the equation by **completing the square**.

Example 9.3.5. Solve the equation $x^2 - 6x + 2 = 0$.

Solution: It becomes apparent rather quickly that factoring this trinomial is not likely to work. Thus we consider forcing a perfect-square trinomial on one side of the equation. Recall that we may complete the square on a quadratic expression of the form $x^2 + bx$ by "adding" $\left(\frac{b}{2}\right)^2$.

$$\begin{aligned} x^2 - 6x + 2 &= 0 \\ x^2 - 6x &= -2 \\ x^2 - 6x + (-3)^2 &= -2 + (-3)^2 \\ x^2 - 6x + 9 &= -2 + 9 \\ (x - 3)^2 &= 7 \\ x - 3 &= \pm\sqrt{7} \\ x &= 3 \pm \sqrt{7} \end{aligned}$$

Example 9.3.6. Solve the equation $3x^2 - 4x - 5 = 0$.

Solution: It again becomes apparent rather quickly that factoring is not a viable option. Thus we consider completing the square, recalling that it is convenient to have a leading coefficient of 1. Hence, we'll divide both sides of the equation by 3 before attempting to complete the square.

$$\begin{aligned}
3x^2 - 4x - 5 &= 0 \\
3x^2 - 4x &= 5 \\
x^2 - \frac{4}{3}x &= \frac{5}{3} \\
x^2 - \frac{4}{3}x + \left(\frac{2}{3}\right)^2 &= \frac{5}{3} + \left(\frac{4}{9}\right) \\
\left(x - \frac{2}{3}\right)^2 &= \frac{19}{9} \\
x - \frac{2}{3} &= \pm\frac{\sqrt{19}}{3} \\
x &= \frac{2}{3} \pm \frac{\sqrt{19}}{3} \\
x &= \frac{2 \pm \sqrt{19}}{3}
\end{aligned}$$

In general, we may solve the equation $ax^2 + bx + c = 0$ by completing the square and extracting square roots as follows:

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c=0 \tag{9.1}$$

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c \tag{9.2}$$

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a} \tag{9.3}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \tag{9.4}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \tag{9.5}$$

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{|2a|} \tag{9.6}$$

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a} \tag{9.7}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \tag{9.8}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{9.9}$$

Note that in moving from equation (6) to equation (7), we replaced $|2a|$ with $2a$. We can justify this as follows: if $a>0$, then $|2a|=2a$. If $a<0$, then $|2a|=-2a$, so we would have $\frac{\pm\sqrt{b^2-4ac}}{-2a}=\frac{\mp\sqrt{b^2-4ac}}{2a}$. But these are the same roots we had to begin with! Thus, we cover both cases ($a>0$ and $a<0$) by just using the $\pm$.

This computation leads to the following theorem and our fourth algebraic solution technique for quadratic equations.

Theorem 9.3.7. *If x is a solution of the equation $ax^2+bx+c=0$ ($a\neq 0$), then*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Logically speaking, this does not mean that $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$ will be solutions, but the reader may verify as an exercise the following theorem.

Theorem 9.3.8. *The values $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ are solutions to the quadratic equation $ax^2+bx+c=0$ where $a\neq 0$.*

Combining these two theorems leads to the theorem known as the Quadratic Formula.

Theorem 9.3.9 (The Quadratic Formula). *The solutions of the quadratic equation in general form* $ax^2 + bx + c = 0$, $a \neq 0$, *are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic expression $ax^2 + bx + c$; it carries a substantial amount of information about the behavior of the function. We will investigate some of its properties after some examples.

Example 9.3.10. Solve each of the following quadratic equations.

1. $x^2 + 4x + 5 = 0$
2. $x^2 + 6x + 9 = 0$
3. $2x^2 - 3x = 5$

Solution.

1.

$$\begin{array}{rcll} x^2 + 4x + 5 & = & 0 & \text{Given} \\ a = 1, b = 4, & \text{and} & c = 5 & \text{Inspection} \\ x & = & \dfrac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} & \text{Quadratic Formula} \\ x & = & \dfrac{-4 \pm \sqrt{-4}}{2} & \text{Simplification} \\ x & = & \dfrac{-4 \pm 2i}{2} & \text{Simplification} \\ x & = & -2 \pm i & \text{Simplification} \end{array}$$

The solutions are $x = -2 + i$ and $x = -2 - i$. Notice that these are complex conjugates of each other.

We may factor the original polynomial as $x^2 + 4x + 5 = (x - (-2 + i))(x - (-2 - i))$.

2.

$$\begin{array}{rcll} x^2 + 6x + 9 & = & 0 & \text{Given} \\ a = 1, b = 6, & \text{and} & c = 9 & \text{Inspection} \\ x & = & \dfrac{-6 \pm \sqrt{6^2 - 4(1)(9)}}{2(1)} & \text{Quadratic Formula} \\ x & = & \dfrac{-6 \pm 0}{2} & \text{Simplification} \\ x & = & -3 & \text{Simplification} \end{array}$$

There is only one zero! Notice that $x^2 + 6x + 9 = (x + 3)^2$; had we factored at the beginning instead of blindly using the quadratic formula, we would have very quickly discovered that $x = -3$ is the only solution.

3.

$$\begin{array}{rcll} 2x^2-3x & = & 5 & \text{Given} \\ 2x^2-3x-5 & = & 0 & \text{Subtract 5} \\ (2x-5)(x+1) & = & 0 & \text{Factoring} \\ 2x-5=0 & \text{or} & x+1=0 & \text{Zero Product Theorem} \\ x=\frac{5}{2} & \text{or} & x=-1 & \text{Add/multiply by equals} \end{array}$$

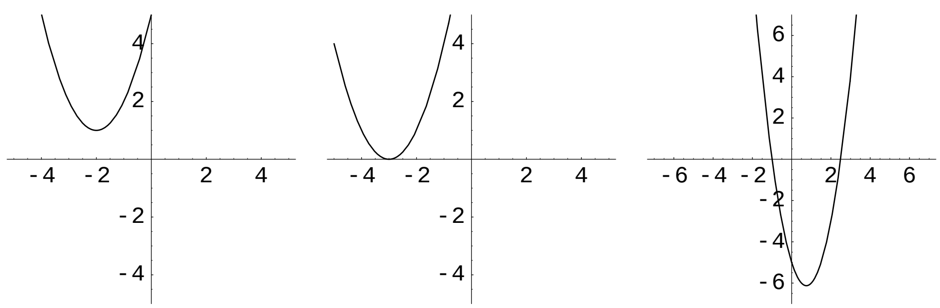
The quadratic formula is not always the best tool; always be on the lookout for easier methods.

For practice, we will also present a solution using the quadratic formula.

$$\begin{array}{rcll} 2x^2-3x & = & 5 & \text{Given} \\ 2x^2-3x-5 & = & 0 & \text{Subtract 5} \\ a=2, b=-3, & \text{and} & c=-5 & \text{Inspection} \\ x & = & \dfrac{-(-3)\pm\sqrt{(-3)^2-4(2)(-5)}}{2(2)} & \text{Quadratic Formula} \\ x & = & \dfrac{3\pm\sqrt{49}}{2(2)} & \text{Simplification} \\ x & = & \dfrac{3\pm 7}{4} & \text{Simplification} \\ x=\frac{10}{4} & \text{or} & x=-\frac{4}{4} & \text{Simplification} \\ x=\frac{5}{2} & \text{or} & x=-1 & \text{Simplification} \end{array}$$

We may factor our polynomial as $2x^2-3x-5=(2x-5)(x+1)$.

We have the following graphs:



$f(x)=x^2+4x+5$ $\quad$ $f(x)=x^2+6x+9$ $\quad$ $f(x)=2x^2-3x-5$

Consider the behavior of the discriminant in each example above. In the first example, $b^2-4ac<0$, and we had two different nonreal complex solutions. This makes sense, since we must take the square root of the discriminant.

In the second example, the discriminant was zero, and we found only one solution, which was real. Again, this makes sense, since the reason we might get two solutions is that we take plus *or* minus the square root of the discriminant. However, if the discriminant is zero, adding it or subtracting it will give the same result, so there is only one solution.

In the third example, the discriminant was positive, and there were two distinct real roots. This, too, is reasonable since we both add and subtract the square root of the discriminant, which will be a real number as long as the discriminant is positive.

The third example also illustrates a general principle: the solutions will be rational if and only if the discriminant is a perfect square (of a rational number). In this case, as in the example, the original quadratic would have factored with rational coefficients. Thus, you can use the discriminant to tell you whether or not to look for a factorization.

In summary, we have the following characterizations of the two solutions to the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac < 0$, then the quadratic equation has no real solutions (and the two complex solutions are complex conjugates), and the graph of the corresponding parabola does not intersect the x-axis.

- If $b^2 - 4ac = 0$, then the quadratic equation has one REAL (repeated) solution, and the graph of the corresponding parabola is tangent to the x-axis at the vertex.

- If $b^2 - 4ac > 0$, then the quadratic equation has two distinct REAL solutions, and the graph of the corresponding parabola intersects the x-axis in two distinct points.

Example 9.3.11. Simplify $\dfrac{5x^2 + 17x + 12}{x^2 + x + 1}$.

Solution: Before we go to the trouble of factoring, let's check the discriminant of the denominator: $1^2 - 4(1)(1) = -3 < 0$, so $x^2 + x + 1$ will not factor over the rationals. Therefore, this quotient is already simplified.

Some quadratic functions may also be "disguised."

Example 9.3.12. Consider $x^4 - 4x^2 - 5 = 0$. This is not a quadratic equation in x, but it *is* quadratic in x^2: $(x^2)^2 - 4(x^2) - 5 = 0$.

$x^4 - 4x^2 - 5$	$=$	0	Given
$(x^2)^2 - 4(x^2) - 5$	$=$	0	Laws of Exponents
$u^2 - 4u - 5$	$=$	0	Substitute $u = x^2$
$(u - 5)(u + 1)$	$=$	0	Factor
$u - 5 = 0$	or	$u + 1 = 0$	Zero Product Theorem
$u = 5$	or	$u = -1$	Add equals
$x^2 = 5$	or	$x^2 = -1$	Substitute $u = x^2$
$x = \pm\sqrt{5}$	or	$x = \pm i$	Extract square root

Example 9.3.13. Solve $x + \sqrt{x} = 1$.

Solution:

$$
\begin{aligned}
x+\sqrt{x} &= 1 && \text{Given}\\
x+\sqrt{x}-1 &= 0 && \text{Subtract 1}\\
u^2+u-1 &= 0 && \text{Substitute } u=\sqrt{x}\\
u &= \frac{-1\pm\sqrt{1-4(1)(-1)}}{2} && \text{Quadratic Formula}\\
u &= \frac{-1\pm\sqrt{5}}{2} && \text{Simplification}\\
\sqrt{x} &= \frac{-1\pm\sqrt{5}}{2} && \text{Substitute } u=\sqrt{x}\\
x &= \left(\frac{-1+\sqrt{5}}{2}\right)^2 && \text{Square both sides}\\
x &= \frac{3\pm\sqrt{5}}{2} && \text{Simplification}
\end{aligned}
$$

Note that $\frac{3+\sqrt{5}}{2}$ is clearly not a solution since it (and its square root) are both greater than one. One may verify that $x=\frac{3-\sqrt{5}}{2}$ is in fact a solution. Be on the lookout for quadratics in disguise!

We close this section with an application of quadratic equations.

Example 9.3.14. If Roy and Joe work together, they can mow a field in 4 hours. Roy working alone will take 3 hours longer than Joe would working alone. How long would each take to mow the field alone?

Solution: Let r_1 and r_2 be the rates of Roy and Joe, respectively. Then their combined rate is r_1+r_2. If we multiply this rate by the time it takes to complete the mowing, we will have one complete job: $(r_1+r_2)4=1$. Also, if t is how long it takes Joe alone, then Roy will need $t+3$ hours to complete the same amount of work, so $r_2t=r_1(t+3)=1$. Therefore, $r_2=\frac{1}{t}$ and $r_1=\frac{1}{t+3}$. This gives us

$$4\left(\frac{1}{t}+\frac{1}{t+3}\right)=1.$$

Multiply both sides by $t(t+3)$ to clear denominators. The resulting equation (after distributing and simplifying) is

$$4(t+3)+4t=t(t+3)$$

or $8t+12=t^2+3t$. Thus we have $t^2-5t-12=0$. This has solutions $t=\frac{5\pm\sqrt{73}}{2}$. Only the positive root makes sense in this context, so Joe will take approximately 6.77 hours on his own, and Roy will take about 9.77 hours alone.

Exercises

Solve each equation by completing the square.

1. $x^2 + 6x + 2 = -1$
2. $x^2 - 5x - 5 = 0$
3. $3x^2 + 8x + 20 = 14$
4. $-4x^2 - 5x = 22$

Solve each equation by using the quadratic formula.

5. $x^2 + 5x - 8 = 0$
6. $x^2 + x + 1 = 0$
7. $-3x^2 - 2x + 4 = 0$
8. $\frac{3}{5}x^2 + 2x - 1 = \frac{4}{5}$

Solve each equation by an appropriate method.

9. $x^2 - 12 = 4$
10. $(x-3)^2 = 5$
11. $x^2 + 8x + 12 = 0$
12. $3x^2 - 5x - 1 = 14$
13. $x^2 + 2x = 4$
14. $3x^2 - 12x + 14 = 2$
15. $(x-3)(x-5) = (x-5)$
16. $x^8 - 6x^4 - 9 = 0$
17. $x^5 - 3x^3 + \frac{1}{4}x = 0$
18. $2x - \sqrt{x} = 1$
19. $\left(\frac{x+2}{x-1}\right)^2 - \frac{x+2}{x-1} - 2 = 0$
20. The sum of a number and its multiplicative inverse is -1. What are the possibilities for the number?
21. If it takes Azuka and Barney 7 hours to paint a house together, and it takes Barney 14 hours to paint the house by himself, how long will it take Azuka working alone?
22. An arrow is shot into the air; its height is given by $h(t) = -4.9t^2 + 30t + 2$ meters above the ground.
    (a) Determine the maximum height of the arrow and when it occurs.
    (b) When does the arrow hit the ground?
    (c) At what times is the arrow 40 meters in the air? Why is there more than one answer?
23. Verify that $(1 + \sqrt{2})$ is a square root of $3 + 2\sqrt{2}$.

9.4 Polynomial Functions and Equations

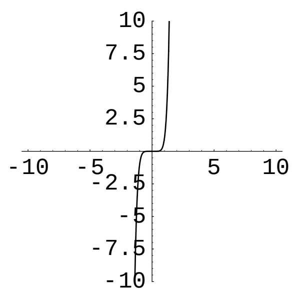
A general polynomial function is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

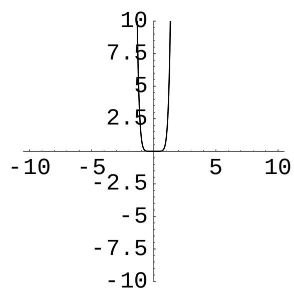
where $a_n \neq 0$. In this case, f has degree n, written $\deg f = n$. The numbers $a_0, a_1, \ldots, a_n$ are called **coefficients**, $a_n x^n$ is the **leading term**, and a_n is the **leading coefficient**. We have already considered some special cases: constant functions (degree 0), linear functions (degree 1), and quadratic functions (degree 2).

The graph of a polynomial is a smooth curve with no sharp turns or breaks. The monomial x^n "looks like" x^2 if n is even and x^3 if n is odd. The difference is that when $n > 3$, the graph will be flatter on $[-1, 1]$ and steeper on $(-\infty, -1) \cup (1, \infty)$.

Example 9.4.1.



$y = x^7$



$y = x^8$

One can determine the shape of a more general polynomial's graph by considering the leading term. For example, consider the polynomial $f(x) = -3x^8 - 4x^5 + 6x^3 - 2$. If x is an extremely large positive number, then $-3x^8$ will be an enormous negative number. By comparison, $-4x^5, 6x^3$, and -2 will all be "small" numbers. Take $x = 1000$. This is not a particularly "large" x, but it will serve to illustrate the idea. We get

$$-3(1000)^8 = -3000000000000000000000000,$$

and

$$f(1000) = -2999999996000006000000002.$$

The relative sizes of these are nearly the same! This idea is summarized in the theorem below.

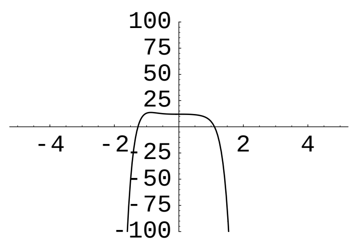
Theorem 9.4.2 (Leading Coefficient Test). *The polynomial $f(x) = a_n x^n + \ldots + a_0$ has the following end behavior.*

1. *If n is even and $a_n > 0$, then the end behavior is $(\nwarrow, \nearrow)$. (Like x^2)*

   If n is even and $a_n < 0$, then then the end behavior is $(\swarrow, \searrow)$. (Like $-x^2$)

2. *If n is odd and* $a_n > 0$, *then then the end behavior is* $(\swarrow, \ \nearrow)$. *(Like* x^3*)*
   If n is odd and $a_n < 0$, *then then the end behavior is* $(\nwarrow, \ \searrow)$. *(Like* $-x^3$*)*

Example 9.4.3. Consider the graph of $f(x) = -3x^8 - 4x^3 + 12$. The theorem tells us that this should have the shape of an inverted "U" if we zoom out far enough. The graph below bears this out.



Such an analysis can tell you, for example, whether your graphing calculator has its window set appropriately.

We now come to an extremely important theorem about dividing polynomials. Its statement may not look familiar, but it is very likely that you are familiar with its application.

Theorem 9.4.4 (The Division Algorithm). *If* $f(x)$ *and* $p(x)$ *are polynomials, then there are unique polynomials* $q(x)$ *and* $r(x)$ *such that* $f(x) = p(x)q(x) + r(x)$, *where* $r(x) = 0$ *or* $0 \leq \deg r(x) < \deg p(x)$.

Example 9.4.5. Divide $f(x) = x^4 - 3x^2 + 2x + 1$ by $x^3 - 4$.

Solution: We employ polynomial long division. Notice that we must insert $0x^3$ into $f(x)$ as a place holder, in the same way that we use a 0 in 108 as a place holder. Also, recall the strategy of polynomial long division: all it really is is a bookkeeping system.

$$
\begin{array}{rrrrrrrrrr}
 & x & & & & & & & & \\
\cline{2-10}
x^3 - 4 \;\big) & x^4 & + & 0x^3 & - & 3x^2 & + & 2x & + & 1 \\
 & x^4 & & & & & - & 4x & & \\
\cline{2-10}
 & & & & - & 3x^2 & + & 6x & + & 1
\end{array}
$$

Thus, in terms of the division algorithm, $f(x) = (x^3 - 4)(x) + (-3x^2 + 6x + 1)$, with $p(x) = (x^3 - 4), q(x) = x$, and $r(x) = -3x^2 + 6x + 1$. The letter q stands for "quotient," and the letter r stands for "remainder."

How do we know when to stop our long division process? The division algorithm tells us we can quit as soon as our remainder has degree less than that of p. In this example, p has degree 3, so we stop when we get to $-3x^2 + 6x + 5$, which has degree 2.

The same reasoning says that if we divide a polynomial by a linear polynomial, the remainder will be a constant.

Example 9.4.6. Divide $x^2 - 3x + 5$ by $x + 2$.

$$\begin{array}{r|rrrr} & x & - & 5 & \\ \hline x+2 & x^2 & - 3x & + & 5 \\ & x^2 & + 2x & & \\ \hline & & - 5x & + & 5 \\ & & - 5x & - & 10 \\ \hline & & & & 15 \end{array}$$

Thus the quotient is $x - 5$ and the remainder is 15. In terms of the division algorithm, we have $f(x) = (x + 2)(x - 5) + 15$.

It is reasonable to ask whether the constant has any particular significance; in fact, it does.

Theorem 9.4.7 (Remainder Theorem). *If $f(x)$ is divided by $x - a$, the remainder is $f(a)$.*

Proof. By the division algorithm, we may write $f(x) = (x - a)q(x) + R$, where $R = 0$ or R has degree 0; in either case, R is a constant (a specific number that doesn't depend on x). Now evaluate both sides of this equation at $x = a$: $f(a) = (a - a)q(a) + R = R$. □

Example 9.4.8. We may verify this by considering again Example 9.4.6. We have $x + 2$ as a divisor. The Remainder Theorem has as hypothesis that f is divided by $x - a$, so in order to use the theorem, we must first write $x + 2$ in this form. We have $x + 2 = x - (-2)$. Thus $a = -2$ (not 2). Now evaluate: $f(-2) = (-2)^2 - 3(-2) + 5 = 15$, the same as our remainder!

The following theorem is closely related to the Remainder Theorem.

Theorem 9.4.9 (Factor Theorem). *Let f be a polynomial. Then $x - a$ is a factor of f if and only if $f(a) = 0$.*

Proof. First, suppose that $x - a$ is a factor of $f(x)$. Then $f(x) = (x - a)q(x)$, with no remainder. Therefore, $f(a) = (a - a)q(x) = 0$.

Next, suppose that $f(a) = 0$. By the Remainder Theorem, if $f(x)$ is divided by $x - a$, the remainder is $f(a)$; in this case, 0. Therefore we may write $f(x) = (x-a)q(x)+0 = (x-a)q(x)$, so $x - a$ is a factor of $f(x)$. □

Example 9.4.10. In Example 9.4.6, $x + 2$ is not a factor of $f(x) = x^2 - 3x + 5$ since $f(-2) = 15 \neq 0$. We already knew this since we performed polynomial long division and found a nonzero remainder; however, if we had had the Factor Theorem, we would not have needed to do the long division to determine this.

Example 9.4.11. Suppose that we wish to factor the polynomial $f(x) = x^4+4x^3-x^2-16x-12$. It does not take long to decide that $x = -1$ is a zero of f, so the Factor Theorem tells us that $x + 1$ will be a factor of $f(x)$. Therefore, it is worth our while to divide $f(x)$ by $x + 1$.

$$
\begin{array}{rrrrrrrrrrr}
 & & x^3 & + & 3x^2 & - & 4x & - & 12 & & \\
\cline{2-11}
x+1 & \big) & x^4 & + & 4x^3 & - & x^2 & - & 16x & - & 12 \\
 & & x^4 & + & x^3 & & & & & & \\
\cline{3-11}
 & & & & 3x^3 & - & x^2 & - & 16x & - & 12 \\
 & & & & 3x^3 & + & 3x^2 & & & & \\
\cline{5-11}
 & & & & & - & 4x^2 & - & 16x & - & 12 \\
 & & & & & - & 4x^2 & - & 4x & & \\
\cline{6-11}
 & & & & & & & - & 12x & - & 12 \\
 & & & & & & & - & 12x & - & 12 \\
\cline{8-11}
 & & & & & & & & & & 0
\end{array}
$$

We see that the remainder is 0, as promised. We now have

$$f(x) = (x+1)(x^3 + 3x^2 - 4x - 12).$$

Notice that $q(x) = x^3 + 3x^2 - 4x - 12$ has degree 3, one less than the degree of f.

We may now factor q by grouping:

$$q(x) = x^2(x+3) - 4(x+3) = (x^2 - 4)(x+3) = (x-2)(x+2)(x+3).$$

Therefore, $f(x) = (x-2)(x+1)(x+2)(x+3)$. As a bonus, we can see that f has exactly four zeros: $2, -1, -2$, and -3.

Notice that we did not have to use polynomial long division repeatedly; once we found one factor, we were able to factor the rest by other methods. Be on the lookout for such things!

Also, it is not a coincidence that f had four zeros and that its degree is four. We have the following theorem.

Theorem 9.4.12. *A polynomial function of degree $n > 0$ has at most n real zeroes.*

Proof. We prove the theorem by mathematical induction. We take as S_n the statement "A polynomial function of degree $n > 0$ has at most n real zeros." Our goal is to prove that S_n is a true statement for all $n \in \mathbb{N}$.

Base Case: If $n = 1$, we have $f(x) = ax + b$ for some $a \neq 0$. Therefore, if $f(x) = 0$, then $ax + b = 0$, so $x = -\frac{b}{a}$. We see that we can have at most one zero if f has degree 1. Thus, S_1 is a true statement.

Induction Step: We may assume that S_k is a true statement for some $k \geq 1$; that is, any polynomial of degree k has at most k real zeros. We must show that S_{k+1} is also a true statement.

Thus, let $f(x)$ be a polynomial of degree $k+1$. If f does not have any real zeros, then f certainly does not have more than $k+1$ zeros, so S_{k+1} is a true statement. If f does have a real zero, say a, then we may write $f(x) = (x - a)q(x)$, where $q(x)$ has degree k. (We need $(x - a)q(x)$ to have degree $k + 1$.) Now, if b is a zero of f, then by the Zero Product Theorem, we must have either $b - a = 0$ or $q(b) = 0$. Since q has at most k real zeros by the induction hypothesis and $x - a$ has at most one real zero by the base case, f can have at most $k + 1$ real zeros, completing the proof.

□

We have developed several number systems in the course of this text, but most have what may be considered a serious "flaw." Consider:

1. The polynomial $f(x) = 2x + 4$ has natural numbers for coefficients, but its only zero is **not** a natural number. We need more numbers to solve the equation $2x + 4 = 0$.
2. If we allow integer coefficients (and solutions) instead of only natural numbers, we can solve $2x + 4 = 0$, but we still cannot solve $2x - 1 = 0$ in the integers. We need still a larger number system to solve this equation.
3. If we move into the rational numbers, we can then solve $2x - 1 = 0$, but we are still unable to solve the equation $x^2 - 2 = 0$ with rational numbers even though the coefficients are rational. (Recall that $\sqrt{2}$ is not rational.)
4. If we allow all real numbers as coefficients and as solutions, we can solve $x^2 - 2 = 0$, but we cannot solve $x^2 + 1 = 0$ in the real numbers, although the coefficients are real numbers. We need yet again a larger number system in order to solve this equation.
5. We will now allow any complex number to be a coefficient or solution. Is this number system large enough that every polynomial equation having complex coefficients has a complex zero? There is no obvious reason for this to be the case; nevertheless, it is the case.

Theorem 9.4.13 (Fundamental Theorem of Algebra). *Every nonconstant polynomial with complex coefficients has at least one (complex) zero.*

NOTE: Remember that $\mathbb{R} \subseteq \mathbb{C}$, so polynomials with real coefficients will also have (possible nonreal complex) zeros, as well.

Proof. The first proof of this important and remarkable theorem is due to Karl Friedrich Gauss, as are many other important theorems. Its proof is beyond the scope of this course; if you are interested in seeing a proof, you will find one in any course in Complex Analysis. □

In fact, we can go a little farther with this result.

Theorem 9.4.14. *A polynomial of degree n has exactly n zeros, some of which may be repeated.*

Proof. This may also be proved by induction; we leave the proof as an exercise. □

What does it mean for a zero to be repeated?

Definition 9.4.15. If a linear factor $x - a$ occurs m times, then a is a zero of **multiplicity** m.

Example 9.4.16. $(x-3)^4(3x+2)(5x-12)^5$ has solution set $\{3, -\frac{2}{3}, \frac{12}{5}\}$. However, 3 has multiplicity 4, $-\frac{2}{3}$ has multiplicity 1, and $\frac{12}{5}$ has multiplicity 5. We could also say that 2 has multiplicity 0.

Exercises

Determine the end behavior of each polynomial. Also determine the maximum possible number of real zeros.

1. $p(x) = x^2 + 5$
2. $f(x) = 2x - 7$
3. $x(t) = -12t^5 - 7t^3 + 2t - 1$
4. $g(x) = 5x^4 + 3x^3 - x - 100000000001$

Express each quotient in terms of the division algorithm. (You may need to perform polynomial long division in order to do this.)

5. $(x^5 - 3x^4 - 2x^3 + x^2 + 4x + 1) \div (x^3 - 2)$
6. $(2x^4 - x - 1) \div (x - 1)$
7. $(3x^3 - 4x^2 + 2x + 2) \div (3x + 3)$
8. $(3x^3 - 7x^2 + x - 2) \div (x^2 + 3)$

Determine the remainder in each case *without* performing long division, and whether the divisor is a factor of the dividend.

9. $(2.4x^3 + x^2 - 3.2x) \div (x - 1.1)$
10. $(2x^6 + 6x^5 - 3x^4 - 9x^3 - 4x - 12) \div (x + 3)$
11. $(4x^4 - 3x^3 - x - 4) \div (x + 4)$
12. $(-2x^5 - 8x^3 - x + 12) \div (x + 2)$
13. $(x^4 - 2x^3 - 8x^2 - 3x + 2) \div (x + 1)$
14. $(-4x^2 - 2x - 4) \div (x - 2)$

Determine the multiplicity of each zero for each polynomial.

15. $(x-2)(x+3)^4(3x-5)^3$
16. $16(x+2)^4(3x-1)^{13}$
17. $(x+1)(x+2)^2(x+3)^3(x+4)^4$
18. Use Mathematical Induction to prove Theorem 9.4.14.

9.5 Rational Roots

In this section, we explore some techniques for finding zeros of polynomials. From the results of the previous section, we know that we can find zeros by factoring completely. We summarize those results in the following theorem.

Theorem 9.5.1. *Let $a \in \mathbb{R}$ and let f be a polynomial function. The following are equivalent.*

1. *$x = a$ is a zero of f.*
2. *$x = a$ is a solution of $f(x) = 0$.*
3. *$(x - a)$ is a factor of the polynomial $f(x)$.*
4. *$(a, 0)$ is an x-intercept of the graph of f.*

Example 9.5.2. Find all real zeroes of $f(x) = x^4 + x^3 - x^2 + x - 2$.

Solution: We can see that $x = 1$ is a zero, so according to the theorem above, $x - 1$ must be a factor. We get $f(x) = (x - 1)(x^3 + 2x^2 + x + 2)$ from polynomial long division. Now we can factor by grouping, and we see that $f(x) = (x - 1)(x + 2)(x^2 + 1)$. Since $x^2 + 1$ does not factor over $\mathbb{R}$, the real roots are $x = 1, -2$. However, $x^2 + 1$ does factor over $\mathbb{C}$ as $x^2 + 1 = (x + i)(x - i)$, so we also have the nonreal complex zeros i and $-i$. We have $f(x) = (x - 1)(x + 2)(x - i)(x + i)$.

How did we know that 1 is a zero? We have a technique that we will discuss below. First, we introduce a simplified division algorithm for when the divisor is of the form $x - a$. This process is called **synthetic division**.

Example 9.5.3. Divide $4x^4 - 7x^3 + x - 5$ by $x + 2$. NOTE: we must have the divisor in the form $x - a$, so $a = -2$. Just as polynomial long division is really a bookkeeping system, so is synthetic division. We will compare the two methods in this example. The major difference between the two is that in polynomial long division, we use the powers of x (x^4, x^3, etc.) to keep track of like terms. In synthetic division, we use only the coefficients, and keep track of like terms according to their **positions** in an array, just as we use in our decimal number system.

We set up both processes in a similar way:

$$x + 2 \;\overline{\big)\; 4x^4 \;-\; 7x^3 \;+\; 0x^2 \;+\; x \;-\; 5} \qquad\qquad -2 \;\overline{\big)\; 4 \quad -7 \quad 0 \quad 1 \quad -5}$$

Notice that in both cases we must include the 0 that holds the place of the x^2 term. It is especially important that we do this in synthetic division, since it is the *position* that tells us the corresponding power of x.

It is also important to notice that in polynomial long division, we have $x + 2$, whereas in synthetic division, we have a -2. We will need to account for this change in sign in the synthetic division process.

In ordinary long division, we keep track of our quotient above the line over the dividend; in synthetic division, we will keep track of the quotient at the bottom, instead. We start the long division process with a $4x^3$; correspondingly, in synthetic division, we record only the coefficient 4. **Note:** This is only valid if the divisor has the form $x - a$. If the divisor looks like $cx - a$, this method cannot be used!

$$
\begin{array}{r|rrrrrrrrr}
 & 4x^3 & & & & & & & & \\
\hline
x+2 \;) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5
\end{array}
\qquad
\begin{array}{r|rrrrr}
 & & & & & \\
\hline
-2\;) & 4 & -7 & 0 & 1 & -5 \\
 & & & & & \\
\hline
 & 4 & & & &
\end{array}
$$

In long division, we next record $4x^3(x+2) = 4x^4 + 8x^3$ in the second row. Where did the $8x^3$ come from? It came from multiplying the $4x^3$ by the 2; the coefficient is obtained by multiplying the 2 in $x+2$ by the 4 in $4x^3$. In synthetic division, we have recorded a -2 from $x+2 = x-(-2)$ and a 4 from $4x^4$. We will multiply these and record the product (-8) in the second row.

$$
\begin{array}{r|rrrrrrrrr}
 & 4x^3 & & & & & & & & \\
\hline
x+2 \;) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5 \\
 & 4x^4 & + & 8x^3 & & & & & &
\end{array}
\qquad
\begin{array}{r|rrrrr}
 & & & & & \\
\hline
-2\;) & 4 & -7 & 0 & 1 & -5 \\
 & & -8 & & & \\
\hline
 & 4 & & & &
\end{array}
$$

Notice the difference in sign for the two 8's. In polynomial long division, we *subtract* the $4x^4 + 8x^3$ from the previous line. In order to account for that sign difference in synthetic division, we *add* the 8 to the previous line.

$$
\begin{array}{r|rrrrrrrrr}
 & 4x^3 & & & & & & & & \\
\hline
x+2 \;) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5 \\
 & 4x^4 & + & 8x^3 & & & & & & \\
\hline
 & & & -15x^3 & + & 0x^2 & + & x & - & 5
\end{array}
\qquad
\begin{array}{r|rrrrr}
 & & & & & \\
\hline
-2\;) & 4 & -7 & 0 & 1 & -5 \\
 & & -8 & & & \\
\hline
 & 4 & -15 & & &
\end{array}
$$

Notice that in both cases we end up with a -15 recorded. In the long division version, we next record a $-15x^2$ above the top bar, but in synthetic division, we already have the -15 recorded at the bottom. In long division, we multiply that $-15x^2$ by $x+2$ and get $-15x^3 - 30x^2$, which we write underneath the last line. Where did the $-30x^2$ come from? It came from multiplying the $-15x^2$ by the 2; accordingly, in synthetic division, we multiply the -15 by the -2, and record that in the second line.

$$
\begin{array}{r|rrrrrrrrr}
 & 4x^3 & - & 15x^2 & & & & & & \\
\hline
x+2 \;) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5 \\
 & 4x^4 & + & 8x^3 & & & & & & \\
\hline
 & & - & 15x^3 & + & 0x^2 & + & x & - & 5 \\
 & & - & 15x^3 & - & 30x^2 & & & &
\end{array}
\qquad
\begin{array}{r|rrrrr}
 & & & & & \\
\hline
-2\;) & 4 & -7 & 0 & 1 & -5 \\
 & & -8 & 30 & & \\
\hline
 & 4 & -15 & & &
\end{array}
$$

Again, we subtract in polynomial long division, and add in synthetic division:

$$
\begin{array}{rrrrrrrrrr}
 & 4x^3 & - & 15x^2 & & & & & & \\
\cline{2-10}
x+2 \,\big) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5 \\
 & 4x^4 & + & 8x^3 & & & & & & \\
\cline{2-10}
 & & - & 15x^3 & + & 0x^2 & + & x & - & 5 \\
 & & - & 15x^3 & - & 30x^2 & & & & \\
\cline{3-10}
 & & & & & 30x^2 & + & x & - & 5
\end{array}
\qquad
\begin{array}{r|rrrrr}
-2 & 4 & -7 & 0 & 1 & -5 \\
 & & -8 & 30 & & \\
\hline
 & 4 & -15 & 30 & &
\end{array}
$$

In long division, we now record a $30x$ on the top line; this 30 is already in the bottom line of our synthetic division. We multiply this $30x$ by $x+2$ to get $30x^2+60x$. Similarly, in synthetic division, we multiply the -2 by the 30 (getting -60) and record that in the second line. We subtract the bottom line from the previous in long division, and add the top two lines in synthetic division, as we have been doing.

$$
\begin{array}{rrrrrrrrrr}
 & 4x^3 & - & 15x^2 & + & 30x & & & & \\
\cline{2-10}
x+2 \,\big) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5 \\
 & 4x^4 & + & 8x^3 & & & & & & \\
\cline{2-10}
 & & - & 15x^3 & + & 0x^2 & + & x & - & 5 \\
 & & - & 15x^3 & - & 30x^2 & & & & \\
\cline{3-10}
 & & & & & 30x^2 & + & x & - & 5 \\
 & & & & & 30x^2 & + & 60x & & \\
\cline{6-10}
 & & & & & & - & 59x & - & 5
\end{array}
\qquad
\begin{array}{r|rrrrr}
-2 & 4 & -7 & 0 & 1 & -5 \\
 & & -8 & 30 & -60 & \\
\hline
 & 4 & -15 & 30 & -59 &
\end{array}
$$

Again, notice that the -59's match. Finally, we record a $-59x$ on the top line for our long division (again observing that the -59 is already recorded in our synthetic division), multiply by $x+2$ (multiplying -59 by -2 in our synthetic division), and subtract (add in our synthetic division).

$$
\begin{array}{rrrrrrrrrr}
 & 4x^3 & - & 15x^2 & + & 30x & - & 59 & & \\
\cline{2-10}
x+2 \,\big) & 4x^4 & - & 7x^3 & + & 0x^2 & + & x & - & 5 \\
 & 4x^4 & + & 8x^3 & & & & & & \\
\cline{2-10}
 & & - & 15x^3 & + & 0x^2 & + & x & - & 5 \\
 & & - & 15x^3 & - & 30x^2 & & & & \\
\cline{3-10}
 & & & & & 30x^2 & + & x & - & 5 \\
 & & & & & 30x^2 & + & 60x & & \\
\cline{6-10}
 & & & & & & - & 59x & - & 5 \\
 & & & & & & - & 59x & - & 108 \\
\cline{7-10}
 & & & & & & & & & 103
\end{array}
\qquad
\begin{array}{r|rrrrr}
-2 & 4 & -7 & 0 & 1 & -5 \\
 & & -8 & 30 & -60 & 108 \\
\hline
 & 4 & -15 & 30 & -59 & 103
\end{array}
$$

Polynomial long division gives us the result

$$(4x^4 - 7x^3 + x - 5) \div (x+2) = 4x^3 - 15x^2 + 30x - 59 + \frac{103}{x+2};$$

that is, in terms of the division algorithm,

$$4x^4 - 7x^3 + x - 5 = (x+2)(4x^3 - 15x^2 + 30x - 59) + 103.$$

What does synthetic division tell us? We read the coefficients of the quotient from the bottom row:

$$(4x^4 - 7x^3 + x - 5) \div (x+2) = 4x^3 - 15x^2 + 30x - 59 + \frac{103}{x+2},$$

the same thing we had before! How do we know what power of x to begin with? We must begin with a power on x one less than the degree of the dividend, since we are dividing by a polynomial of degree one. After that, the powers simply descend until we reach the last entry, which is the remainder.

Example 9.5.4. Divide $5x^3 - 7x^2 + 2x$ by $x - 3$ using synthetic division.

Solution: First, we write this in the form

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \end{array}$$

to get things started. Remember, when the divisor has the form $x - a$, we record a itself as the first entry in the first row. In this case, we have $x - 5$, so we record the 5. Next, we bring down the 3.

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & & & \\ \hline & 3 & & & \end{array}$$

Now multiply that 3 by the 5, and record that in the second row of the next column.

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & 15 & & \\ \hline & 3 & & & \end{array}$$

Add the -7 and the 15:

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & 15 & & \\ \hline & 3 & 8 & & \end{array}$$

Multiply the 8 by the 5, and record the result in the second row of the next column.

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & 15 & 40 & \\ \hline & 3 & 8 & & \end{array}$$

Add the 2 and the 40:

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & 15 & 40 & \\ \hline & 3 & 8 & 42 & \end{array}$$

Multiply the 42 by the 5, and record the result in the second row of the next column.

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & 15 & 40 & 210 \\ \hline & 3 & 8 & 42 & \end{array}$$

Finally, add the 210 and the 0.

$$\begin{array}{r|rrrr} 5 & 3 & -7 & 2 & 0 \\ & & 15 & 40 & 210 \\ \hline & 3 & 8 & 42 & 210 \end{array}$$

Therefore, we have a quotient of $3x^2 + 8x + 42$ with a remainder of 210, which we may also write in any of the equivalent forms

$$\frac{3x^3 - 7x^2 + 2x}{x-5} = 3x^2 + 8x + 42 + \frac{210}{x-5},$$

$$(3x^3 - 7x^2 + 2x) \div (x-5) = 3x^2 + 8x + 42 + \frac{210}{x-5},$$

or

$$3x^3 - 7x^2 + 2x = (x-5)(3x^2 + 8x + 42) + 210.$$

The last, of course, is written in terms of the division algorithm.

Notice that a is a zero if and only if that last entry in row 3 is a zero; this is also an easy way to determine whether a given number is a zero! Of course, we have to have a number to check to begin with, so we need some system for finding possible zeros. We have one in the following theorem.

Theorem 9.5.5 (Rational Root Test). *Let f be a polynomial of degree $n \geq 1$ with integer coefficients, say $f(x) = a_n x^n + \ldots + a_0$, $a_n \neq 0$. Then the rational zeros of f have the form $\frac{p}{q}$, where p is a factor of a_0 and q is a factor of a_n.*

That is, the only possible rational zeroes are of the form $\dfrac{\text{factors of constant}}{\text{factors of leading coefficient}}$.

That's powerful! We can use this theorem to find a (relatively) short list of potential rational zeros.

Note: Henceforth, we will just write the final result of our synthetic division, rather than listing each step separately.

Example 9.5.6. Find all rational zeros of $f(x) = 4x^4 - 4x^3 + 5x^2 - 4x + 1$.

Solution: The leading coefficient is 4, which has factors $\pm 1, \pm 2$, and ± 4. The constant term is 1, which has factors ± 1. Therefore, the potential rational zeros are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$. We now apply synthetic division to determine which are actual zeros.

First, notice that the signs alternate; this means that no negative number can be a zero. Consider: if $a > 0$, then $-a < 0$, and $4(-a)^4 - 4(-a)^3 + 6(-a)^2 - 4(-a) + 1 = 4a^4 + 4a^3 + 5a^2 + 4a + 1 > 0$. Therefore, we need only check $1, \frac{1}{2}$, and $\frac{1}{4}$. We begin with $x = 1$.

$$
\begin{array}{r|rrrrr}
1 & 4 & -4 & 5 & -4 & 1 \\
  &   & 4 & 0 & 5 & 1 \\
\hline
  & 4 & 0 & 5 & 1 & 2
\end{array}
$$

Thus $x = 1$ is not a zero (since division by $x - 1$ leaves a nonzero remainder). Next, we try $x = \frac{1}{2}$.

$$
\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & 5 & -4 & 1 \\
  &   & 2 & -1 & 2 & -1 \\
\hline
  & 4 & -2 & 4 & -2 & 0
\end{array}
$$

Therefore, $x = \frac{1}{2}$ is a zero, and $x - \frac{1}{2}$ is a factor of f. We have

$$f(x) = \left(x - \frac{1}{2}\right)(4x^3 - 2x^2 + 4x - 2).$$

We may factor the quotient by grouping, but it will be more instructive at this point to show how the method proceeds.

We are reduced to trying to find the zeros of $4x^3 - 2x^2 + 4x - 2$. Notice that any zero of $4x^3 - 2x^2 + 4x - 2$ is automatically a zero of f, so we need only look among the potential rational zeros we already have for f. It is not necessary to try $x = 1$ again, since we already know that $x = 1$ is not a zero of f, but we do need to try $x = \frac{1}{2}$ again, as it may be a repeated root.

$$
\begin{array}{r|rrrr}
\frac{1}{2} & 4 & -2 & 4 & -2 \\
  &   & 2 & 0 & 2 \\
\hline
  & 4 & 0 & 4 & 0
\end{array}
$$

Thus we now have $4x^3 - 2x^2 + 4x - 2 = (x - \frac{1}{2})(4x^2 + 4)$, which means that

$$\begin{aligned} f(x) &= (x - \frac{1}{2})^2(4x^2 + 4) \\ &= (x - \frac{1}{2})^2(4)(x^2 + 1) \\ &= [(x - \frac{1}{2})^2(2^2)](x^2 + 1) \\ &= [(x - \frac{1}{2})(2)]^2(x^2 + 1) \\ &= (2x - 1)^2(x^2 + 1). \end{aligned}$$

The zeros of $x^2 + 1$ are $\pm i$, as can be seen by considering $x^2 = -1$ or using the quadratic formula to solve $x^2 + 1 = 0$. Finally, we arrive at

$$f(x) = (2x - 1)^2(x - i)(x + i).$$

The rational zero of f is $x = \frac{1}{2}$ (with multiplicity 2), and $x = i$ and $x = -i$ are nonreal complex zeros of f.

Notice that not all of the potential rational zeros actually were zeros, and that not all of the zeros were rational. Also, it is worth noting that synthetic division lends itself very nicely to this process; it takes far less time, effort, and space than polynomial long division. BE WARNED: synthetic division **only** works with divisors of the form $x - a$; if you try it with something else, it will fail.

Exercises

Perform synthetic division where appropriate. If it is not appropriate, perform polynomial long division. In either case, give the remainder.

1. $(3x^3 - 5x^2 - 3x - 1) \div (x - 2)$
2. $(5x^5 - 3x^4 + 2x^2 - x - 2) \div (x + 3)$
3. $(2x^4 + x^3 - x^2 - 5x - 7) \div (2x + 3)$
4. $(x^6 - 3x^4 + 3x^2 - 1) \div (x + 1)$
5. $(-4x^5 + 5x^2 + 8) \div (x^2 + 1)$
6. $(6x^8 - 7x^5 + 4x^4 + 9x^3 + 5x - 2) \div (x - 4)$
7. Answer each of the following for the polynomial $f(x) = 3x^3 - 2x^2 + 6x - 4$. Give justification for **each** answer.

   (a) How many zeros does f have?

(b) How many real zeros does f have?

(c) List all the possible rational zeros of f.

(d) Find all the zeros of f.

(e) Classify each zero as rational, irrational, or nonreal complex.

(f) Write f as a product of linear factors.

8. Answer each of the following for the polynomial $f(x) = 3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16$. Give justification for **each** answer.

   (a) How many zeros does f have?

   (b) How many real zeros does f have?

   (c) List all the possible rational zeros of f.

   (d) Find all the zeros of f.

   (e) Classify each zero as rational, irrational, or nonreal complex.

   (f) Write f as a product of linear factors.

Find the zeros of each polynomial. Classify each zero as rational, irrational, or nonreal complex, and express f as a product of linear factors. Also give the multiplicity of each zero.

9. $x^4 + x^3 - 2x^2 + 4x - 24$

10. $x^6 - 4x^5 - x^4 + 20x^3 - 20x^2$

11. $x^7 + 2x^6 - 3x^5 - 6x^4 + 3x^3 + 6x^2 - x - 2$

12. $x^5 + 5x^4 + 11x^3 + 16x^2 + 18x + 12$

13. $2x^4 + 3x^3 - 21x^2 - 33x - 11$

14. $12x^4 - 46x^3 + 12x^2 + 20x - 8$