Solutions to Homework Assignment 15

MATH 141-01

Section 3.8, Page 260

1, 2, 7, 11, 13, 19, 23, 23, 27, 31, 32, 34, 44

1. Since $V(x) = x^3$, $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$ by the Chain Rule.

2. (a) Since $A(r) = \pi r^2$, $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$.

   (b) Using (a), we have $\left.\dfrac{dA}{dt}\right|_{r=30} = 2\pi(30)(1) = 60\pi\text{m}^2/\text{s}$.

7. $\dfrac{dy}{dt} = 3x^2\dfrac{dx}{dt} + 2\dfrac{dx}{dt} = 3(2)^2(5) + 2(5) = 70$.

11. We are given the altitude and speed of the plane, and we want to find how fast it is going away from the radar station on the ground. If the distance on the ground from the radar station to directly under the plane is x and the distance from the radar station to the plane is r, then $x^2 + 1 = r^2$. Thus $2x\dfrac{dx}{dt} = 2r\dfrac{dr}{dt}$. When $r = 2$, we have $x^2 + 1 = 4$, so $x = \sqrt{3}$. We get $\dfrac{dr}{dt} = \dfrac{\sqrt{3}}{2}(500) = 250\sqrt{3}$. See the back of the book for a picture.

13. We are given the heights of the pole and the man, the man's speed, and a distance from the pole. We are to find how fast the end of the shadow he casts is moving.
    Let x be the distance from the pole to the end of the shadow, and let y be the distance from the pole to the man. We have similar triangles, and $\dfrac{15}{6} = \dfrac{x}{x-y}$. (The distance from the tip of the shadow to the man is $x - y$.) Thus $15x - 15y = 6x$, so $9x = 15y$. Differentiating gives $9\dfrac{dx}{dt} = 15\dfrac{dy}{dt}$, so $\dfrac{dx}{dt} = \dfrac{15}{9}(5) = \dfrac{25}{3}$ ft/s.

23. We have $A = \dfrac{1}{2}bh$, where A is the area, b the base, and h the height of the triangle. These are all functions of time. Differentiating, which requires the product rule, gives $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\dfrac{dh}{dt} + h\dfrac{db}{dt}\right)$. We know $\dfrac{dA}{dt} = 2$, $\dfrac{dh}{dt} = 1, h = 10$, and $A = 100$, and we can calculate $b = \dfrac{2A}{h} = 20$. Substitution gives us $2 = \dfrac{1}{2}\left(20(1) + 10\dfrac{db}{dt}\right)$, so $\dfrac{db}{dt} = -1.6\text{cm/min}$.

19. Let $w(t)$ be the volume of water in the tank at time t, and let c be the constant rate at which water is being pumped into the tank. We are told that $\dfrac{dw}{dt} = c - 10000\ \text{cm}^3/\text{min}$. A cross-section of the conical tank (that passes through its vertex) is a triangle, and the filled part makes a similar triangle. If h is the height of the water at time t and r is the radius, then $\dfrac{h}{r} = \dfrac{6}{2} = 3$.
    The volume of the water is $w(t) = \dfrac{1}{3}\pi r^2 h = r^3$. Since we are given $\dfrac{dh}{dt}$, we will eliminate r from this: $r = \dfrac{h}{3}$, so $w(t) = \dfrac{\pi}{27}h^3$. Now $\dfrac{dw}{dt} = \dfrac{\pi}{9}h^2\dfrac{dh}{dt}$. At the time we are interested in, this is $\dfrac{\pi}{9}(200)^2(20) = \dfrac{800000\pi}{9} = c - 10000$. Thus $c = 10000 + \dfrac{\pi 800000}{9} \approx 289,300\ \text{cm}^3/\text{min}$. (Watch those units!)

27. The volume of the pile is $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\cdot\dfrac{1}{4}\pi h^3$ in this case. We get $\dfrac{dV}{dt} = \dfrac{\pi}{4}h^2\dfrac{dh}{dt} = 30$ when $h = 10$, so $\dfrac{dh}{dt} = \dfrac{120}{\pi 10^2} \approx 0.382$ ft/min.

31. Differentiate both sides with respect to time: $V\dfrac{dP}{dt} + P\dfrac{dV}{dt} = 0$. Substitution gives $600(20) + 150\dfrac{dV}{dt} = 0$, so $\dfrac{dV}{dt} = -\dfrac{1200}{150} = 80\ \text{cm}^3/\text{min}$.

32. Differentiate both sides with respect to time: $V^{1.4}\frac{dP}{dt} + 1.4V^{0.4}\frac{dV}{dt}P = 0$. Substituting the known quantities gives $400^{1.4}(-10) + 1.4(400)^{0.4}\frac{dV}{dt}(80) = 0$, so $\frac{dV}{dt} = \frac{400^{1.4}(10)}{1.4(400)^{0.4}(80)} \approx 35.7\ \text{cm}^3/\text{min}$.

44. Let x be the distance between the ends of the hands. Recall the law of cosines: if a, b, and c are the lengths of the sides of a triangle and θ is the angle between the sides a and b, then $c^2 = a^2+b^2-2ab\cos\theta$. (This collapses to the Pythagorean Theorem if $\theta = \frac{\pi}{2}$.) In our situation, $c = x, a = 8$, and $b = 4$, while θ is variable.
Let θ be the angle between the hands. The minute hand travels one full revolution in an hour, or 2π radians per hour. The hour hand travels only 1/12 as fast, or $\frac{2\pi}{12} = \frac{\pi}{6}$ radians per hour. The angle between them at 1:00 is therefore decreasing at the rate of $2\pi - \frac{\pi}{6} = -\frac{11\pi}{6}$ radians per hour.
Since $x^2 = 8^2+4^2-2(8)(4)\cos\theta$, we get $2x\frac{dx}{dt} = 64\sin\theta\frac{d\theta}{dt}$, so $\frac{dx}{dt} = \frac{32\sin\theta}{x}\left(-\frac{11\pi}{6}\right) = -\frac{176\pi\sin\theta}{3x}$.
We need both θ and x at 1:00.
At 1:00, θ is 1/12 of the way around the circle, so $\theta = \frac{\pi}{6}$. Using the law of cosines, we get $x^2 = 64+16-2(8)(4)\cos(\pi/6) = 80-32\sqrt{3}$. Thus $x = \sqrt{80-32\sqrt{3}}$. (Holy cow!)
Now substitution gives us $\frac{dx}{dt} = -\frac{176\pi(1/2)}{3\sqrt{80-32\sqrt{3}}} \approx -18.59$ radians per hour, or 0.0051 radians per second.