# Solutions to Homework Assignment 14

**MATH 141-01**

**Section 3.6, Stewart 6e, Page 233**

**3, 7, 12, 15, 16, 19, 27, 29, 30, 39, 40, 44, 45, 46, 53**

3. (a) $x^{-1}+y^{-1}=1$, so $-x^{-2}-y^{-2}y'=0$, or $-\frac{1}{x^2}-\frac{1}{y^2}y'=0$ Therefore $y'=-\frac{y^2}{x^2}$.

   (b) $\frac{1}{x}+\frac{1}{y}=1$ implies that $xy\left(\frac{1}{x}+\frac{1}{y}\right)=xy$. Thus $y+x=xy, xy-y=x$, and $y=\frac{x}{x-1}$. Differentiation gives $y'=\frac{1(x-1)-1(x)}{(x-1)^2}=-\frac{1}{(x-1)^2}$.

   (c) To compare to (a), we need to substitute $y=\frac{x}{x-1}$ into $y'$: $y'=-\frac{\frac{x^2}{(x-1)^2}}{x^2}=-\frac{1}{(x-1)^2}$, which is what we found in (b).

1.

7. Differentiate both sides with respect to $x$: $2x+(y+xy')-2yy'=0$. Note that we had to use the product rule on the second term. Now $(x-2y)y'=-2x-y$, so $y'=\frac{-2x-y}{x-2y}$.

15. Differentiate both sides with respect to $x$: $e^{x^2y}(2xy+x^2y')=1+y'$. Note here that I had to use the chain rule on the exponential and the product rule on its exponent. Also, differentiating $y$ with respect to $x$ yields $y'$. Now solve for $y'$: $2xye^{x^2y}+x^2e^{x^2y}y'=1+y'$, so $x^2e^{x^2y}y'-y'=1-2xye^{x^2y}$. Now $y'(x^2e^{x^2y}-1)=1-2xye^{x^2y}$ and $y'=\frac{1-2xye^{x^2y}}{x^2e^{x^2y}-1}$.

16. We have $(x+y)^{1/2}=1+x^2y^2$. Differentiate both sides with respect to $x$: $\frac{1}{2}(x+y)^{-1/2}(1+y')=2xy^2+2x^2yy'$, which can be rewritten as $\frac{1}{2\sqrt{x+y}}+\frac{1}{2\sqrt{x+y}}y'=2xy^2+2x^2yy'$. Now $\frac{1}{2\sqrt{x+y}}-2xy^2=\left(2x^2y-\frac{1}{2\sqrt{x+y}}\right)y'$. Therefore, we arrive at

$$y'=\frac{\frac{1}{2\sqrt{x+y}}-2xy^2}{2x^2y-\frac{1}{2\sqrt{x+y}}}=\frac{1-4xy^2\sqrt{x+y}}{4x^2y\sqrt{x+y}-1},$$

where I have multiplied numerator and denominator by $2\sqrt{x+y}$.

27. $2x+2yy'=2(2x^2+2y^2-x)(4x+4yy'-1)$. At $(0,1/2)$, this becomes $0+y'=2(0+2(1/2)^2-0)(0+2y'-1)$, or $y'=2y'-1$. Thus $y'=1$. An equation is therefore $y-\frac{1}{2}=x-0$, or $y=x+\frac{1}{2}$.

29. $4(x^2+y^2)(2x+2yy')=25(2x-2yy')$. At $(3,1)$, this is $4(9+1)(6+2y')=25(6-2y')$. Thus $240+80y'=150-50y'$, so $130y'=-90$ and $y'=-\frac{9}{13}$. Our equation is $y-1=-\frac{9}{13}(x-3)$, or $y=-\frac{9}{13}x+\frac{40}{13}$.

39. This refers to the problem above. If the tangent is horizontal, then $y'=0$, so we have $4(x^2+y^2)(2x)=25(2x)$. This is satisfied if $x=0$, and $x=0$ gives $2y^4=-25y^2$, forcing $y=0$. However, at $(0,0)$, we cannot solve for $y'$. (Our equation just becomes $0=0$!) A quick glance at the graph tells us that there is no tangent line at $(0,0)$ since the graph meets itself there, so this is okay. Therefore, $x\neq 0$ and we may divide both sides by $2x$ to obtain $4(x^2+y^2)=25$. This means that the points on the circle $x^2+y^2=\frac{25}{4}$ and on the lemniscate give us horizontal tangent lines. Substitute $\frac{25}{4}$ into the original equation in place of $x^2+y^2$ to get $2(25/4)^2=25(x^2-(25/4-x^2))$. (I also substituted $25/4-x^2$ for $y^2$.) This becomes $\frac{25}{8}=2x^2-\frac{25}{4}$, or $\frac{75}{8}=2x^2$. Thus $x^2=\frac{75}{16}$ and $x=\pm\frac{5\sqrt{3}}{4}$. (Clean livin'!)

    Now we need the values of $y$ that go with these. We know that $y^2=25/4-x^2=\frac{25}{4}-\frac{75}{16}=\frac{25}{16}$, so

$y = \frac{5}{4}$.
Whew!

40. $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$, so $y' = -\frac{2b^2x}{2a^2y}$. At $(x_0, y_0)$, this is $y' = -\frac{b^2x_0}{a^2y_0}$. Now $y - y_0 = -\frac{b^2x_0}{a^2y_0}(x - x_0)$. Multiply both sides by $y_0$ and divide both sides by $b^2$: $\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = -\frac{xx_0}{a^2}x + \frac{x_0^2}{a^2}$, so $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, as desired.

45. $4x + 2yy' = 0$, so $y' = -\frac{2x}{y}$ for the first curve. For the second, $1 = 2yy'$, so $y' = \frac{1}{2y}$. These curves meet when $x = y^2$ and $2x^2 + y^2 = 3$, so $2(y^2)^2 + y^2 = 3$. Thus $2y^4 + y^2 - 3 = 0$. This gives $(2y^2+3)(y^2-1) = 0$. $2y^2 + 3 = 0$ has no solutions, but $y^2 = 1$ has the solutions $y = \pm 1$, both of which give $x = 1$. At $(1, -1)$, the first curve has slope $y' = -\frac{2}{-1} = 2$ and the second has $y' = \frac{1}{2}$, and these are negative reciprocals of each other.
At $(1, 1)$, the first curve has slope $-2$ and the second has slope $\frac{1}{2}$, so again the slopes are negative reciprocals. Thus the two curves are orthogonal since their tangent lines are perpendicular where the curves intersect.

46. The first curve has $2x + 2yy' = a$, giving $y' = \frac{a - 2x}{2y}$, and the second has $2x + 2yy' = by'$, giving $y' = -\frac{2x}{2y - b}$. The product of these slopes is

$$\begin{aligned}\frac{-2ax + 4x^2}{4y^2 - 2by} &= \frac{4x^2 - 2ax}{4(by - x^2) - 2by} \\ &= \frac{4x^2 - 2ax}{2by - 4x^2} \\ &= \frac{4x^2 - 2ax}{2ax - 4x^2},\end{aligned}$$

which is $-1$, as desired. Note that the curves (in fact they are circles) meet at the origin, where their tangents are vertical and horizontal, respectively.

53. We are to find the intersection of the two tangent lines shown. $x^2 + 4y^2 = 5$ gives $2x + 8yy' = 0$, so $y' = -\frac{x}{4y}$. Let $(a, b)$ be the point of tangency of the upper tangent line. The slope of this line is $\frac{b - 0}{a - (-5)} = \frac{b}{a + 5}$. It is also $-\frac{a}{4b}$, so these must be equal: $\frac{b}{a+5} = -\frac{a}{4b}$, so $4b^2 = -a^2 - 5a$. This implies that $a^2 + 4b^2 = -5a$, but we already know that $a^2 + 4b^2 = 5$ since $(a, b)$ is on the ellipse. Thus $a = -1$. Now $4b^2 = 4$, so $b = \pm 1$. Since we are on the top half, $b = 1$. Therefore, $y' = -\frac{-1}{4(1)} = \frac{1}{4}$. The tangent line therefore has equation $y = \frac{1}{4}(x + 5)$, so at $x = 3$, $y = \frac{1}{4}(8) = 2$. This is the height of the lamp.