

Solutions to Homework Assignment 22

MATH 141-01

Section 4.7, Page 262

2, 5, 11, 14, 16, 17, 23, 31, 36, 59, 64

2. Let x and y be the numbers. We wish to minimize xy subject to $x - y = 100$. Thus $y = x - 100$, so $f(x) = x(x - 100) = x^2 - 100x$. $f'(x) = 2x - 100$ has its only zero at $x = 50$. Since $f''(x) = 2$ is always positive, we must have a local minimum, which will also be a global minimum since there is only the one critical number. Thus $x = 50$ and $y = -50$.
5. Let x and y be the dimensions of the rectangle. We wish to maximize $A = xy$ subject to $2x + 2y = 100$. Thus $y = 50 - x$, so $A(x) = x(50 - x) = 50x - x^2$. $A'(x) = 50 - 2x$, so the only critical number is $x = 25$. Since $A''(x) = -2$, A is concave down everywhere; thus, $x = 25$ gives a local and global maximum. We get $y = 25$, so the maximum area occurs for a 25 by 25 square.
11. Assume that the field is to be x feet by y feet. If we assume that the dividing fence is parallel to the side of length x , then the amount of fence the farmer needs is $3x + 2y$, and this is what we wish to minimize. (Minimizing the length of fence will also minimize the cost of the fence.) Our constraint is $xy = 1,500,000$, so $y = \frac{1500000}{x}$. We get $f(x) = 3x + \frac{3000000}{x}$, so $f'(x) = 3 - \frac{3000000}{x^2}$. The only critical number is at $x = 1000$, and since $f''(x) = \frac{6000000}{x^3}$ is positive at this point, we have a local (and global) minimum. The y -dimension is 1500 feet.
16. (a) Let the given area be A , and let the dimensions be x and y . Then $P = 2x + 2y$ is what we want to minimize, subject to $xy = A$. Thus $y = \frac{A}{x}$, so $P = 2x + \frac{2A}{x}$. This gives $P'(x) = 2 - \frac{2A}{x^2}$, which is zero for $x^2 = A$. Thus $x = \sqrt{A}$ and $y = \sqrt{A}$ at this point. Since $P''(x) = \frac{4A}{x^3}$ is positive here, we have a local (and global) minimum. Notice that $x = y$, so we do have a square.
- (b) Let the given perimeter be P , and let the dimensions be x and y . Then $2x + 2y = P$, so $y = \frac{P}{2} - x$. The function we want to maximize is $A(x) = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$. Now $A'(x) = \frac{P}{2} - 2x$ has its only zero at $x = \frac{P}{4}$. Since $A''(x) = -2$, this must be a local maximum. But for $x = \frac{P}{4}$, $y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$, so we again have a square.
17. Let (x, y) be a point on the line. The distance from (x, y) to the origin (which is $(0, 0)$) is $\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$. If we minimize the square of the distance, we also minimize the distance, so we will work with $x^2 + y^2$ instead. Since (x, y) lies on the line $y = 4x + 7$, we are actually minimizing $f(x) = x^2 + (4x + 7)^2$. $f'(x) = 2x + 2(4x + 7)(4) = 34x + 56$, so $x = -\frac{28}{17}$ is the only critical number. $f''(x) = 34 > 0$, so we must have a local (and global) minimum here. The y -coordinate is $4\left(-\frac{28}{17}\right) + 7 = \frac{7}{17}$. Thus, the closest point is $(-28/17, 7/17)$.
23. The height of the triangle is $\frac{\sqrt{3}}{2}L$. Let x be the width of the rectangle and let y be the height. We get similar triangles: $\frac{\sqrt{3}L/2}{L/2} = \frac{\sqrt{3}L/2 - y}{x/2}$, or $\sqrt{3} = \frac{\sqrt{3}L - 2y}{x}$. Solving for x gives $x = \frac{\sqrt{3}L - 2y}{\sqrt{3}}$. The area of the rectangle is $A(y) = \frac{\sqrt{3}L - 2y}{\sqrt{3}}y = \frac{\sqrt{3}Ly - 2y^2}{\sqrt{3}}$. Thus $A'(y) = \frac{\sqrt{3}L - 4y}{\sqrt{3}}$, which is only zero for $y = \frac{\sqrt{3}L}{4}$. Since $0 \leq y \leq \sqrt{3}L$, we compare $A(0) = 0$, $A(\sqrt{3}L/4) = \frac{\sqrt{3}L^2}{8}$, and $A(\sqrt{3}L/2) = 0$. Our maximum area occurs at $y = \sqrt{3}L/4$, for which $x = L/2$.
31. Let x be the width and y be the height of the poster. Then $(x - 8)(y - 12) = 384$, so $y = \frac{384}{x - 8} + 12$. The

area of the whole poster is $A(x) = x \cdot \left(\frac{384}{x-8} + 12 \right) = \frac{384x}{x-8} + 12x$. We get $A'(x) = \frac{384(x-8) - 384x}{(x-8)^2} + 12 = \frac{-3072}{(x-8)^2} + 12$. This is zero if $(x-8)^2 = 256$, so $x-8 = \pm 16$. Only the positive root makes sense here, so $x = 24$ cm. Also, $A''(x) = \frac{3072(2)}{(x-8)^3} > 0$ for $x > 8$, so this is a local minimum. The y -dimension is $y = 36$.

59. Let P be the given perimeter, and let y be the base and x the length of each of the other two sides. Then $2x + y = P$, so $y = P - 2x$. The height of the triangle can be found from the Pythagorean Theorem: $(y/2)^2 + h^2 = x^2$, so $h = \sqrt{x^2 - (P/2 - x)^2}$. The area is thus $A(x) = \frac{1}{2}(P - 2x)\sqrt{Px - P^2/4}$. We

get $A'(x) = \frac{1}{2} \left(-2\sqrt{Px - P^2/4} + \frac{1}{2\sqrt{Px - P^2/4}} P(P - 2x) \right) = \frac{1}{4\sqrt{Px - P^2/4}} (-6Px + 2P^2)$. This

is zero for $x = P/3$ and undefined for $x = P/4$. At $x = P/4$, the two upper sides make up half the perimeter; for the third side to also be half, we must have a flat triangle. That is, $x = P/4$ is an endpoint of the domain. The other endpoint is $x = P/2$. We compare these: $A(P/4) = 0$, $A(P/3) = \frac{1}{12\sqrt{3}}P^2$, and $A(P/2) = 0$. Thus, our maximum is at $x = P/3$, so $y = P/3$ as well and the triangle is equilateral.

64. Let $x = QR$. Let $h_1 = QP$ and $h_2 = TS$. (Note that h_1 and h_2 are constant.) If we call the distance QT d , then $RT = d - x$. Let $L_1 = PR$ and $L_2 = SR$; we wish to minimize $L(x) = L_1 + L_2 = \sqrt{x^2 + h_1^2} + \sqrt{(d-x)^2 + h_2^2}$. We get $L'(x) = \frac{x}{\sqrt{x^2 + h_1^2}} - \frac{d-x}{\sqrt{(d-x)^2 + h_2^2}}$. This is zero when the two

expressions are equal. The really, really cool part is that the first expression is $\cos \theta_1$ and the second is $\cos \theta_2$! Thus, our critical point occurs when the cosines are equal. Since both θ_1 and θ_2 are between 0 and $\pi/2$, they must in fact be equal.

At the endpoints, the values are $L'(0) = \frac{-d}{\sqrt{d^2 + h_2^2}} < 0$ and $L'(d) = \frac{d}{\sqrt{d^2 + h_1^2}}$, so we change from decreasing to increasing at the critical point. Therefore, we have a minimum there.