Solutions to Homework Assignment 18

MATH 141

Section 4.2, Stewart 6e, Page 219

1, 2, 5, 9, 12, 14, 17, 18, 20, 21, 23, 26, 34

1. $f(1) = -4 = f(3)$. Since f is a polynomial, it is differentiable and continuous everywhere, including $[1, 3]$. Therefore, Rolle's theorem applies and $f'(c) = 0$ for some $c \in (1, 3)$.
   Specifically, $f'(x) = 2x - 4$, and this is zero for $x = 2$. Thus, $c = 2$.

5. $f(-1) = 0 = f(1)$, but $f'(x) = -\dfrac{2}{3x^{1/3}}$ is never zero on $(-1, 1)$. Although f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(x)$ is undefined at $x = 0$. Thus, there is no contradiction to Rolle's Theorem.

9. See the back of the book for the graphs. For part (c), $f'(x) = 1 - \dfrac{4}{x^2}$. We need to know when this is equal to $\dfrac{f(8) - f(1)}{8 - 1} = 0.5$. We solve $1 - \dfrac{4}{x^2} = \dfrac{1}{2} : 2x^2 - 8 = x^2$, so $x^2 = 8$. Thus $x = \sqrt{8} = 2\sqrt{2}$. (We ignore the negative root since it does not lie in $[1, 8]$.)

14. Since f is a rational function, it is differentiable and continuous on its domain, which includes $[1, 4]$. $f'(x) = \dfrac{x + 2 - x}{(x + 2)^2} = \dfrac{2}{(x + 2)^2}$. The slope of the secant line is $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{2/3 - 1/3}{3} = \dfrac{1}{9}$. We solve $\dfrac{2}{(x + 2)^2} = \dfrac{1}{9}$ for x: $18 = (x + 2)^2$, so $x + 2 = \pm 3\sqrt{2}$ and $x = \pm 3\sqrt{2} - 2$. Only the positive root is in the interval, so $c = 3\sqrt{2} - 2$.

17. $f(-1) = -6$ and $f(0) = 1$, so f has a root x_1 in $(-1, 0)$ by the Intermediate Value Theorem. $f'(x) = 20x^4 + 3x^2 + 2 \geq 2$, so $f'(x)$ is never zero. If f had another root x_2, then Rolle's Theorem would imply that f' had a zero in (x_1, x_2), which is not the case. Therefore, f cannot have a second root, so it has exactly one.

20. If $f(x) = x^4 + 4x + c$, then $f'(x) = 4x^3 + 4 = 4(x + 1)(x^2 - x + 1)$. If f had three distinct zeros, then f' would have to have at least 2 zeros by Rolle's Theorem. However, the only zero of f' is $x = -1$. Therefore, f has at most two distinct zeros.

21. (a) Let P be a polynomial of degree 3. If P has four roots, then between each consecutive pair will lie a zero of the derivative by Rolle's Theorem. But the derivative P' is a polynomial of degree 2; if it has three zeros, then between each consecutive pair will lie a zero of P'', again by Rolle's Theorem applied to P'. This means that p'' will have 2 zeros. Now P'' is a polynomial of degree 1, which we know has at most one zero, so this is not possible. Therefore, P cannot have 4 distinct zeros.

    (b) If our polynomial of degree n is P, then for P to have $n + 1$ zeros would require that P' have at least n zeros, as above. But P' is a polynomial of degree $n - 1$, and we may continue backward as we did in (a) to find that a polynomial of degree 1 would have two zeros.

23. Since f is differentiable on $[1, 4]$, it is automatically continuous on $[1, 4]$, so the Mean Value Theorem applies. Whatever $f(4)$ is, there will be a $c \in (1, 4)$ such that $f'(c) = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{f(4) - 10}{3}$, so $f(4) = 3f'(c) + 10 \geq 3(2) + 16$. Thus, $f(4) \geq 16$.

26. Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$ and differentiable on (a, b) since f and g are. Note that $h(a) = 0$. By the Mean Value Theorem, there is a number $c \in (a, b)$ such that $h'(c) = \dfrac{h(b) - h(a)}{b - a} = \dfrac{h(b)}{b - a} = \dfrac{f'(c) - g'(c)}{b - a}$. Since $f'(c) < g'(c)$, the numerator is negative. Therefore, $h(b) = f'(c) - g'(c) < 0$, so $f(b) - g(b) < 0$. This is what we were hoping for: $f(b) < g(b)$.

34. Let $h(x) = f(x) - x$. If f has two fixed points a and b, then $h(a) = 0 = h(b)$, so h has a derivative of zero somewhere between a and b. But $h'(x) = f'(x) - 1$, which is never zero! Therefore f cannot have two fixed points.