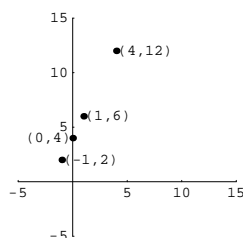


7.4 Graphing Functions

We return to graphing. The power of graphing is in the visual representation it can give of an otherwise mysterious relationship. Given the points $(-1, 2)$, $(4, 12)$, $(0, 4)$, $(1, 5)$, can you see the relationship among them? Probably not! What if we graph them?



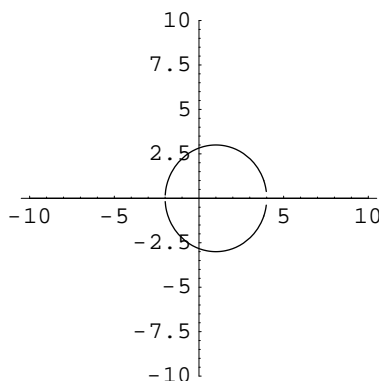
Viewing the graph, it is not hard to believe that these points all lie on a line, although that was not readily apparent from just looking at the ordered pairs. Scientists, business people, journalists, and, in fact, most professions regularly use graphical representations of data to help determine or illustrate relationships.

In this section we will build up a library of basic graphs we should be able to recognize, and then see how they may be transformed. You will want to memorize the graphs of the basic functions so that you can recognize them at a glance.

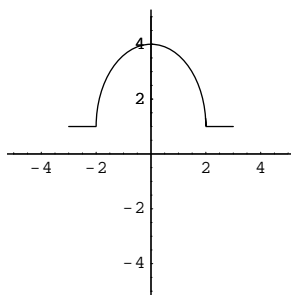
First, let's consider the conditions under which a given graph is the graph of a function $y = f(x)$. Recall that for a relation to be a function we need each x -value in the domain to be paired with exactly one y -value. Since equal x -values correspond graphically to points on a vertical line (consider the graph of $x = 2$), we have the following theorem.

Theorem 7.4.1 (Vertical Line Test). *A graph in the xy -coordinate plane is the graph of a function $y = f(x)$ if and only if every vertical line intersects the graph at most once.*

Example 7.4.2. The relationship defined by the equation $(x - 1)^2 + y^2 = 9$ does not represent a function because the graph of this equation is a circle with center $(1, 0)$ and radius 3, which fails the vertical line test.



Example 7.4.3. Does the following graphical relationship represent a function? If so, find the domain and range of the function.

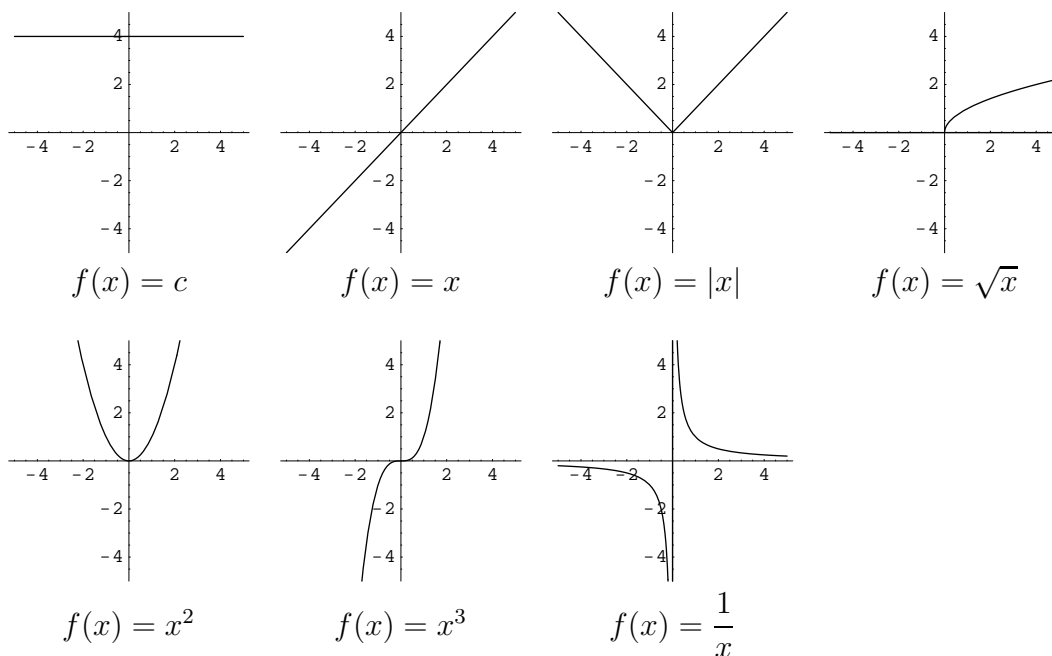


Solution: Since the graph passes the vertical line test, this relationship is a function, f . The domain is the set of x -values used as first coordinates and the range is the set of second coordinates of the ordered pairs comprising f . Hence, the domain appears to be $[-3, 3]$, and the range is $[1, 4]$.

For the vertical line test to be very useful to us, we need to have knowledge of the graph of a given relation. For a completely unknown function, we must plot enough ordered pairs belonging to the function to give us a general idea of what its graph looks like. Consider the following “basic” functions.

$f(x)$	Name	Returns...	Example	Domain
c	Constant	c regardless of x .	$f(-2) = c$	\mathbb{R}
x	Identity	the input.	$f(-2) = -2$	\mathbb{R}
$ x $	Absolute Value	x if $x \geq 0$, and $-x$ if $x < 0$.	$f(-2) = 2$	\mathbb{R}
\sqrt{x}	Square root	$a \geq 0$ such that $a^2 = x$	$f(4) = 2$	$[0, \infty)$
x^2	Square	the square of x	$f(-2) = 4$	\mathbb{R}
x^3	Cube	the cube of x	$f(-2) = -8$	\mathbb{R}
$\frac{1}{x}$	Reciprocal	the multiplicative inverse of x	$f(2/5) = 5/2$	$\{x \in \mathbb{R} x \neq 0\}$

The graphs of these “basic” functions are given below.



We have several transformations we can do with these (and other) functions and still get functions of the same basic shape. Let $c > 0$, and let f be a given function (not necessarily one of those above). We will compare the graph of $f(x)$ to the graph of $g(x) = f(x + c)$.

Remember that in graphing a function, the x -coordinate represents a domain element and the y -coordinate (or “height”) represents the function value at x . Notice that, for example, $f(0) = g(-c)$, so g has the same height at $-c$ as f does at 0. Also, $f(-2) = g(-2 - c)$, so g has the same height at $-2 - c$ as f does at -2 . In general, g will have the same height at $x - c$ as f does at x . Geometrically, this just means that the graph of g follows *exactly the same* “ups and downs” as f does; that is, the graph of g has the same shape as the graph of f . However, while the graph of g attains the same heights as does the graph of f , it does so at $x - c$ instead of at x ; that is, c units to the left of where f does.

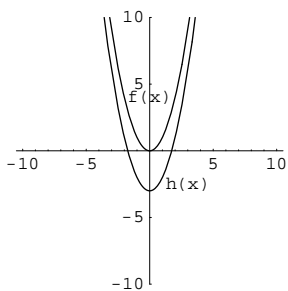
In summary, the graph of $g(x) = f(x + c)$ is exactly the same shape as the graph of f , but translated to the left by c units. Similar arguments lead to the table below.

New function:	Effect on graph of $y = f(x)$
$h(x) = f(x) + c$	Vertical shift up c units
$h(x) = f(x) - c$	Vertical shift down c units
$h(x) = f(x + c)$	Horizontal shift left c units
$h(x) = f(x - c)$	Horizontal shift right c units
$h(x) = -f(x)$	Reflection across x -axis
$h(x) = f(-x)$	Reflection across y -axis
$h(x) = cf(x)$	Vertical stretch by a factor of c for $c > 1$; shrink for $0 < c < 1$
$h(x) = f(cx)$	Horizontal shrink by a factor of c for $c > 1$; stretch for $0 < c < 1$

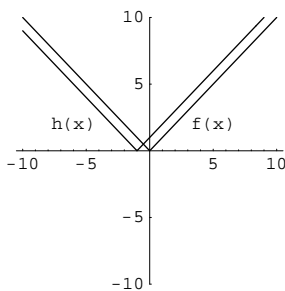
One can see that there are two fundamentally different ways to modify a graph: one can modify the input into f , as $f(x + c)$ or $f(cx)$, or one can modify the output from f , as $f(x) + c$ or $cf(x)$. **The parentheses matter**; read very carefully and consider every symbol.

Example 7.4.4. For each function, identify the basic function and the transformation involved.

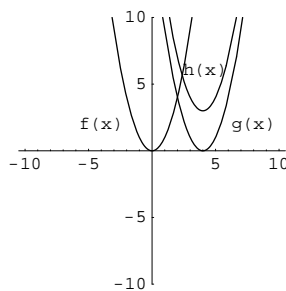
1. $h(x) = x^2 - 3$. The basic function is $f(x) = x^2$; h is a downward shift by 3 units.
2. $h(x) = |x + 1|$. The basic function is $f(x) = |x|$; h is a shift to the left by 1 unit.
3. $h(x) = (x - 4)^2 + 3$. The basic function is $f(x) = x^2$; we have two transformations here. If we put $g(x) = (x - 4)^2$, then $h(x) = g(x) + 3$. To obtain the graph of g , shift the graph of f right 4 units. To obtain the graph of h , shift the graph of g up 3 units.
4. $h(x) = -x^2$. The basic function is $f(x) = x^2$; h is a reflection across the x -axis.
5. $h(x) = 7x^2$. The basic function is $f(x) = x^2$; h is a vertical stretch by a factor of 7.
6. $h(x) = \frac{x^2}{4}$. The basic function is $f(x) = x^2$; h is a vertical shrink by a factor of $1/4$.
7. $h(x) = \sqrt{-x}$. The basic function is $f(x) = \sqrt{x}$; h is a reflection across the y -axis.
8. $h(x) = \sqrt{2x}$. The basic function is $f(x) = \sqrt{x}$. $h(x) = f(2x)$, so its graph is a horizontal compression by a factor of 2 of that of f .
9. $h(x) = -|x| + 3$. The basic function is $f(x) = |x|$. Let $g(x) = -f(x)$, so $h(x) = g(x) + 3$. Therefore, we first reflect the graph of f across the x -axis, and then shift up 3 units.
10. $h(x) = \sqrt{-x + 7} = \sqrt{-(x - 7)}$. The basic function is $f(x) = \sqrt{x}$. Let $g(x) = \sqrt{-x} = f(-x)$. Then the graph of g is the reflection of the graph of f across the y -axis. Also, $h(x) = g(x - 7) = f(-(x - 7)) = \sqrt{-(x - 7)} = \sqrt{-x + 7}$. Therefore, the graph of h is the graph of g shifted right 7 units. We again had to perform two transformations.
11. $h(x) = \frac{-1}{x + 2}$. The basic function is $f(x) = \frac{1}{x}$. Let $g(x) = -f(x)$ (a reflection across the x -axis), and then $h(x) = g(x + 2)$ (a shift right by 2 units).
12. $h(x) = (2x + 6)^3$. The basic function is $f(x) = x^3$. Let $g(x) = f(2x)$. Then $h(x) = g(x + 3)$. Thus the graph of h is the graph of x^3 after a horizontal compression by a factor of 2 and a shift to the left 3 units. We could also have used $g(x) = f(x + 6)$, a shift to the left by 6 units, followed by $h(x) = g(2x) = f(2x + 6)$, a horizontal compression by a factor of 2. We can choose either way, but we must be careful not to mix them together.



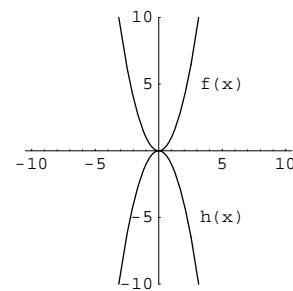
1. $h(x) = x^2 - 3$



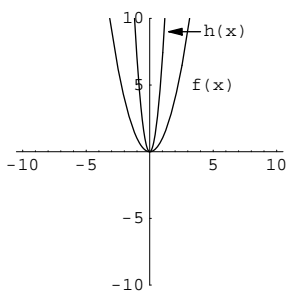
2. $h(x) = |x + 1|$



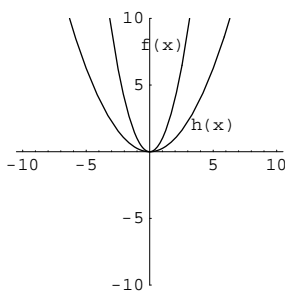
3. $h(x) = (x - 4)^2 + 3$



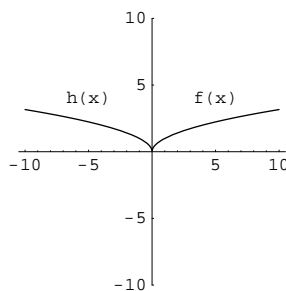
4. $h(x) = -x^2$



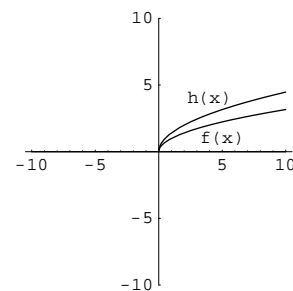
5. $h(x) = 7x^2$



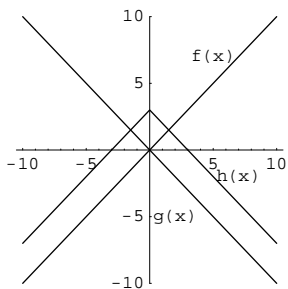
6. $h(x) = \frac{1}{4}x^2$



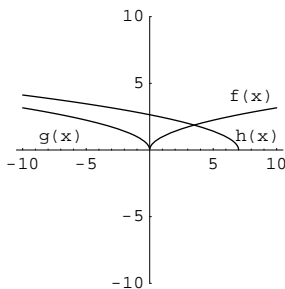
7. $h(x) = \sqrt{-x}$



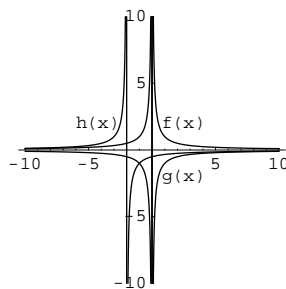
8. $h(x) = \sqrt{2x}$



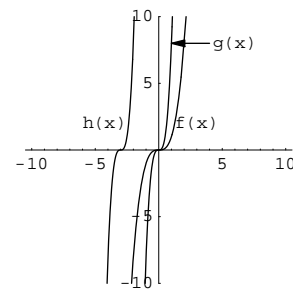
9. $h(x) = -|x| + 3$



10. $h(x) = \sqrt{-x + 7}$



11. $h(x) = \frac{-1}{x + 2}$



12. $h(x) = (2x + 4)^3$

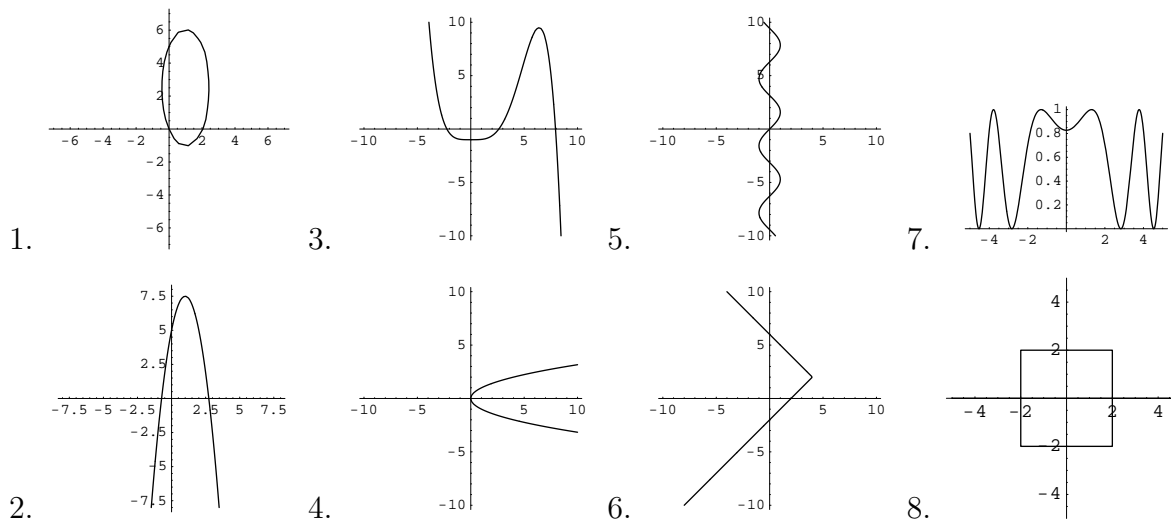
Note that the graph touches the x -axis if and only if the y -coordinate is 0. The corresponding x -value is called a **zero** of f .

Definition 7.4.5. If $f(x) = 0$, then x is a **zero** of f .

Zeros are especially important in equation solving, one of the major purposes of algebra.

Exercises

Determine whether the given graph is the graph of a function.



9. For each graph in problems 1 through 8 that represents a function, determine the domain and range from the graph as well as possible. (You may need to extrapolate beyond what you can see on the graph. Realize, though, that doing so is somewhat hazardous!)
10. For each graph in problem 1 that represents a function f , determine $f(2)$ from the graph, if possible. Realize that this will only be an approximation. If it is not possible, explain why not.

Sketch the graph of each function. If possible, do so by transforming the graph of a known function rather than plotting points.

11. $f(x) = x^2 + 1$

14. $f(x) = \sqrt{2x - 5}$

17. $f(x) = \sqrt{-(x + 1)}$

12. $f(x) = \frac{1}{x - 2}$

15. $f(x) = 2|x + 3| - 1$

18. $f(x) = (3x)^3$

13. $f(x) = \frac{x + 1}{x}$

16. $f(x) = -|x - 2|$

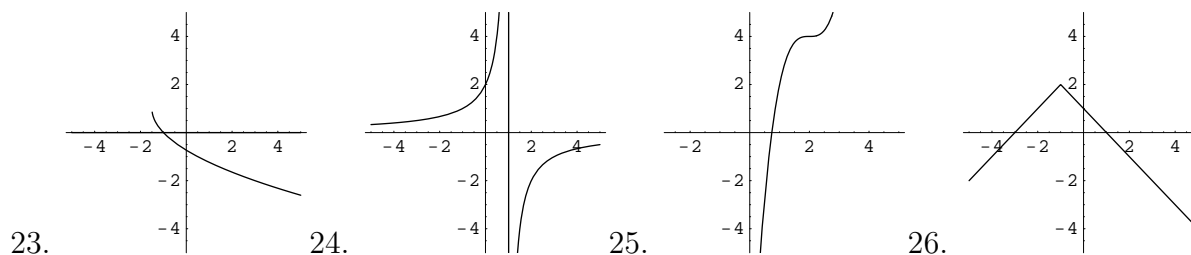
19. $f(x) = -\frac{1}{x - 3} + 2$

20. Convince yourself that each of the transformations in the table “Transformations of Graphs” does what the table claims it will do.

21. The **floor** function is another basic function; $\text{floor}(x)$, also written $\lfloor x \rfloor$ (pronounced “floor x ”) is the greatest integer that is less than or equal to x . For example, $\lfloor 2.75 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, $\lfloor -4.1 \rfloor = -5$, and $\lfloor 461 \rfloor = 461$. Sketch a graph of the floor function.

22. The **ceiling** function is another basic function; $\text{ceil}(x)$, also written $\lceil x \rceil$ (pronounced “ceiling x ”) is the least integer that is greater than or equal to x . For example, $\lceil 2.75 \rceil = 3$, $\lceil \pi \rceil = 4$, $\lceil -4.1 \rceil = -4$, and $\lceil 461 \rceil = 461$. Sketch a graph of the ceiling function.

Each graph below is a transformation of some basic function. Identify the basic function.



27. For each $x \in \mathbb{R} - \mathbb{Z}$, show that $\lceil x \rceil = \lfloor x + 1 \rfloor$.

28. The following data was collected in an experiment in which the height h of a dropped ball was measured (in centimeters) at several times t . Sketch a graph of the height h versus time t , and decide whether the graph is a transformation of one of our basic graphs.

$h(t)$	202	201	200	196	191	185	178	169	159	148	135	122	107	90	73	54	34	12
t	$\frac{3}{60}$	$\frac{5}{60}$	$\frac{7}{60}$	$\frac{9}{60}$	$\frac{11}{60}$	$\frac{13}{60}$	$\frac{15}{60}$	$\frac{17}{60}$	$\frac{19}{60}$	$\frac{21}{60}$	$\frac{23}{60}$	$\frac{25}{60}$	$\frac{27}{60}$	$\frac{29}{60}$	$\frac{31}{60}$	$\frac{33}{60}$	$\frac{35}{60}$	$\frac{37}{60}$