MATH 150

Final Exam **Tuesday, December 8, 2020**

Name: KEY

Remember to **show your work**. Unsupported solutions will receive no credit.

Nothing takes place in the world whose meaning is not that of some maximum or minimum.
– Leonhard Euler

Formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

	Volume	Surface Area
Cylinder	$\pi r^2 h$	$2\pi rh + 2\pi r^2$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Cone	$\frac{1}{3}\pi r^2 h$	$\pi r\ell + \pi r^2$

Computation (60 points)

In this section, you may not compute integrals on your calculator (except to check). You must integrate by hand to receive **any credit at all**.

1. (15 points) Differentiate. You do not need to simplify your answers.

   (a) $f(x) = 5x^6 - 4x^3 + 7x + 2$
   Solution: $f'(x) = 5 \cdot 6x^5 - 4 \cdot 3x^2 + 7 \cdot 2x$

   (b) $f(x) = e^{3x} \sin x$
   Solution: $f'(x) = 3e^{3x} \sin x + (\cos x)e^{3x}$

   (c) $f(x) = \dfrac{\sin x}{3x^2 - 4}$
   Solution: $f'(x) = \dfrac{(\cos x)(3x^2-4) - 6x \sin x}{(3x^2-4)^2}$

2. (5 points) Find an equation of the tangent line to the graph of $f(x) = 2x + \tan(3x) - 1$ at $x = 0$.

   Solution: We need a point and the slope. $f(0) = 2(0) + \tan(3(0)) - 1 = -1$, so $(0, 1)$ is our point. The slope, of course, comes from the derivative: $f'(x) = 2 + 3\sec^2(3x)$, so $f'(0) = 2 + 3\sec^2(0) = 5$. Thus our tangent line has equation $y - 1 = 5(x - 0)$, or $y = 5x + 1$.

3. (5 points) Compute the sum $\sum_{k=2}^{6} \frac{3k}{k+4}$. You do not need to simplify.

Solution: $\sum_{k=2}^{6} \frac{3k}{k+4} = \frac{3(2)}{2+4} + \frac{3(3)}{3+4} + \frac{3(4)}{4+4} + \frac{3(5)}{5+4} + \frac{3(6)}{6+4}$

4. (20 points) Integrate. In the case of definite integrals, you must show all steps; a numerical answer from your calculator is **not** sufficient. (That is, you must convince me you know how to find the definite integrals without your calculator.)

(a) $\int_{-1}^{2} 6x^3 dx$

Solution: $= \frac{6}{4}x^4\Big|_{-1}^{2} = \frac{3}{2}(2^4 - (-1)^4) = \frac{45}{2}$.

(b) $\int \frac{6}{x} + e^{4x} dx$

Solution: $= 6\ln|x| + \frac{1}{4}e^{4x} + C$

(c) $\int_0^1 (x+1)(3x^2+6x+4)^{12} dx$

Solution: $= \frac{1}{6}\int_0^1 (6x+6)(3x^2+6x+4)^{12} dx = \frac{1}{6}\cdot\frac{1}{13}(3x^2+6x+4)^{13}\Big|_0^1 = \frac{1}{78}((3(1)^2+6(1)+4)^{13} - (3(0)^2+6(0)+4)^{13}) = \frac{1}{78}(13^{13} - 4^{13})$.

(d) $\int \sin(x) - \frac{4}{x^2} dx$

Solution: $= \int \sin(x) - 4x^{-2} dx = -\cos(x) - \frac{4}{-1}x^{-1} + C = -\cos(x) + \frac{4}{x} + C$.

5. (5 points) Compute: $\lim_{x\to\infty} \frac{e^{3x}}{e^{3x}+x}$

Solution: This has the form $\frac{\infty}{\infty}$, so by L'Hôpital's Rule it equals $\lim_{x\to\infty} \frac{3e^{3x}}{3e^{3x}} = 1$.

6. (10 points) Compute $\int_1^6 x^2 dx$ using the **definition** of the definite integral.

Solution: First, $\Delta x = \frac{6-1}{n} = \frac{5}{n}$. I will use right endpoints: $x_k = 1 + \frac{5}{n}k$. We get:

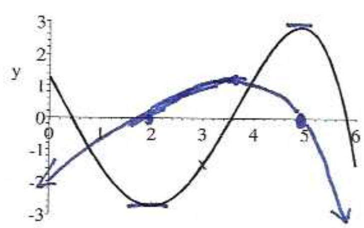
$$
\begin{aligned}
\int_1^6 x^2dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{5}{n}k\right)^2\frac{5}{n}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{10}{n}k+\frac{25}{n^2}k^2\right)\frac{5}{n}\\
&= \lim_{n\to\infty}\frac{5}{n}\left(n+\frac{10}{n}\cdot\frac{n(n+1)}{2}+\frac{25}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right)\\
&= \lim_{n\to\infty}\left(5+\frac{25(n+1)}{n}+\frac{125(n+1)(2n+1)}{6n^2}\right)\\
&= 5+25+\frac{125}{3}\\
&= \frac{215}{3}.
\end{aligned}
$$

Concepts (60 points)

7. (5 points) If $f'(x) = \dfrac{3x^2}{\ln(x)+6}$ and $g(x) = f(x) - 4$, what is $g'(2)$?

   Solution: Since f and g differ by a constant, $g'(x) = f'(x)$. Thus $g'(2) = \dfrac{3(2)^2}{\ln(2)+6}$.

8. (10 points) The graph of $y = f(x)$ is shown below. Sketch a graph of $y = f'(x)$ on the same set of axes.



$y = f(x)$

9. (10 points) Let $f(x) = x^4 - 8x^2 + 3$. Use techniques of calculus to find the absolute extrema of f on $[0, 3]$. (I.e., don't just use your calculator!)

   Solution: We need the critical points. Since f' is defined everywhere, we only need the kind that come from solving $f'(x) = 0$. We have $f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$, so $x = 0, 2$, or -2. Of these, only 0 and 2 are in the given interval. We evaluate f at the relevant critical points and the endpoints of the interval: $f(0) = 3, f(2) = -13$, and $f(3) = 12$, so the absolute max is 12 and occurs at $x = 3$, and the absolute min is -13 and occurs at $= 2$.

10. (5 points) Find the area of the region below the graph of $f(x) = 16 - x^2$ between $x = 0$ and $x = 4$.

    Solution: The area is $\displaystyle\int_0^4 16 - x^2dx = 16x - \frac{x^3}{3}\bigg|_0^4 = 16(4) - \frac{4^3}{3} - 0 = \frac{128}{3}$.

11. (10 points) For $f(x) = \sqrt{x}$, use the tangent line to estimate $\sqrt{9.01}$.

Solution: Since 9.01 is close to 9 (a number we can compute the square root of exactly), I will find an equation of the tangent line at $x = 9$. We have $f(9) = 3$ and $f'(x) = f12\sqrt{x}$, so $f'(9) = \frac{1}{6}$. Thus our tangent line is $y - 3 = \frac{1}{6}(x - 9)$, or $y = \frac{1}{6}(x-9) + 3$. At $x = 9.01$, this becomes $\frac{1}{6}(9.01 - 9) + 3 = \frac{0.01}{6} + 3 \approx 3.00167$.

12. (10 points) Sand drains through an hourglass at the rate of 2cm^3 per minute and forms a conical pile in which the height and radius are always equal. At what rate is the radius of the pile changing when the volume is 9π cm^3?

Solution: With equal radius and height, the volume of the sand pile is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$. I chose to replace the h with r (rather than the other way around) because this question is asking us to relate V and r. Now $\frac{dV}{dt} = \frac{1}{3} \cdot 3r^2 \frac{dr}{dt} = r^2 \frac{dr}{dt}$. When the volume is 9π, r satisfies $9\pi = \frac{1}{3}\pi r^3$, so $27 = r^3$ and $r = 3$. Since we are given the rate $\frac{dV}{dt} = 2$, we arrive at $2 = 3^2 \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{2}{9}$.

13. (10 points) True or False. Assume f is differentiable everywhere.

(a) F (minimum) If $f'(a) = 0$ and $f''(a) > 0$, then $f(a)$ is a local maximum.

(b) T (definition) If $F(x)$ is an antiderivative for $f(x)$, then $F'(x) = f(x)$.

(c) T (MVT) If $f(2) = 10$ and $f(5) = 4$, then $f'(c) = -2$ for some $c \in [2, 5]$.

(d) T (f increasing) If $f'(x) > 0$ for all x, then $f(4)$ is less than $f(7)$.

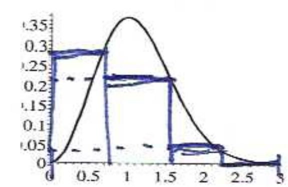
(e) F (CD) A function f is concave up if the slope of f is decreasing.

Applications (30 points)

14. (10 points) The graph below shows the velocity of a proton in m/s after t seconds have elapsed. Estimate how far the proton travels for $t \in [1, 3]$ using 4 subintervals.

I will use right endpoints.

$\Delta x = \frac{3-0}{4} = 0.75$.



$d \approx 0.75(0.28 + 0.22 + 0.04 + 0)$
(Eyeballing the heights)

15. (10 points) In a certain chemical reaction, the concentration C of a chemical is given by $C = \frac{4t}{2t+1}$. The **rate of reaction** is the derivative of C with respect to time, $\frac{dC}{dt}$. Find the rate of reaction at $t = 3$ seconds.

Solution: $\frac{dC}{dt} = \frac{4(2t+1) - 2(4t)}{(2t+1)^2}$. At $t = 3$, this is $\frac{4(7) - 2(12)}{7^2} = \frac{4}{49}$.

16. (10 points) Bilbo is designing barrels for the comfort of anyone who has to ride inside of them. He wants the barrel to be a cylinder, and for storage reasons, its circumference $C = 2\pi r$ plus its length ℓ must be no more than 4m. What is the largest possible volume of a such a barrel?

Solution: We are given that $2\pi r + \ell \leq 4$, and the volume of the cylinder is $V = \pi r^2 \ell$. The bigger we can make r and ℓ, the bigger the volume will be, so we'll get the biggest volume at the extreme $2\pi r + \ell = 4$. Solving for ℓ gives $\ell = 4 - 2\pi r$, and substituting this into V yields

$$V = \pi r^2(4 - 2\pi r) = 4\pi r^2 - 2\pi^2 r^3.$$

Now $V'(r) = 8\pi r - 6\pi^2 r^2 = 2\pi r(4 - 3\pi r)$. The critical points occur when $V' = 0$, so $r = 0$ or $r = \dfrac{4}{3\pi}$. Now $0 \leq r \leq \dfrac{4}{2\pi}$ (max r is when $\ell = 0$), so we compute $V(0) = 0, V(4/(3\pi)) = \pi \left(\dfrac{4}{3\pi}\right)^2 \left(4 - 2\pi \cdot \dfrac{4}{2\pi}\right)$, and $V(4/2\pi) = 0$. The middle one is the max.

Bonus! (10 points)

17. (10 points) The graph of $f'(x)$ is given to the right. Draw a possible graph of $f(x)$ on the same set of axes.