MATH 150

Exam 2Key **Monday, October 12, 2020**

Name:______________________

Remember to **show your work**. Unsupported solutions will receive no credit.

The calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word.
– W. F. Osgood, quoted in Bulletin American Mathematical Society

1. (10 points) Use the **definition** of the derivative to show that $\frac{d}{dx}(4x^2) = 8x$. **You must use the definition and show your work to receive credit!**

   Solution:

$$\begin{aligned}
\frac{d}{dx}(4x^2) &= \lim_{h\to 0} \frac{4(x+h)^2 - 4x^2}{h} \\
&= \lim_{h\to 0} \frac{4(x^2+2xh+h^2) - 4x^2}{h} \\
&= \lim_{h\to 0} \frac{4x^2+8xh+4h^2-4x^2}{h} \\
&= \lim_{h\to 0} \frac{8xh+4h^2}{h} \\
&= \lim_{h\to 0} \frac{h(8x+4h)}{h} \\
&= \lim_{h\to 0} 8x+4h \\
&= 8x.
\end{aligned}$$

2. (15 points) Calculate the derivative of each function. **DO NOT** just write down an answer; I will need to see your steps in order to award credit.

   (a) $f(x) = \sqrt[3]{x}$.

   Solution: $f(x) = x^{1/3}$, so $f'(x) = \frac{1}{3}x^{-2/3}$.

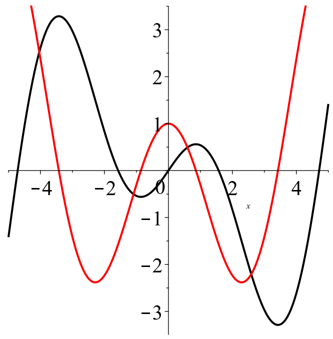
   (b) $f(x) = x^2(6x^4 - 4x^2 + 1)^9$.

   Solution: (Product and Chain Rules) $f'(x) = 2x(6x^4-4x^2+1)^9 + x^2 \cdot 9(6x^4 - 4x^2+1)^8(24x^3-8x)$.

   (c) $f(x) = \frac{3x}{4x^3-8}$

   Solution: (Quotient Rule) $f'(x) = \frac{3(4x^3-8) - 12x^2(3x)}{(4x^3-8)^2}$.

3. (10 points) Below is the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$ on the same set of axes. **NOTE:** a crude sketch is fine; I just want the general shape of the graph of f'.



Solution: The red graph is $y = f'(x)$.

4. (20 points) Let $f(x) = x^3 - 6x^2$.

   (a) Find the critical points and local extrema of f.

   Solution: $f'(x) = 3x^2 - 12x = 3x(x-4)$, so the critical points are at $x = 0$ and $x = 4$. A sign chart would have $x = 0$ and $x = 4$ marked on the number line, dividing the number line into three pieces. Reading from left to right, the signs of f' in those intervals would be $+, -, +$. Thus f has a local maximum of 0 at $x = 0$ and a local minimum of -32 at $x = 4$.

   (b) Find the inflection points of f.

   Solution: $f''(x) = 6x - 12$, which is only 0 for $x = 2$. For $x < 2$, $f''(x) < 0$, and for $x > 2$, $f''(x) > 0$, so f has an inflection point at $x = 2$.

5. (20 points) Suppose that f is a differentiable function with values as in the table.

x	$f(x)$	$f'(x)$
-2	4	-2
0	3	0
1	4	6
3	7	0
4	0	-3

   (a) What are the critical points of f according to the table?

   (b) Identify each critical point as a local maximum, a local minimum, or neither.

   (c) Find an equation of the tangent line to the graph of f at $x = 1$.

   (d) Approximate $f(1.1)$.

Solution: (a) The critical points are where $f'(x) = 0$, so they are at $x = 0$ and $x = 3$.

(b) The sign of f' changes from $-$ to $+$ at $x = 0$, so $x = 0$ gives a local minimum. It changes from $+$ to $-$ at $x = 3$, so $x = 3$ gives a local maximum.

(c) Since $f(1) = 4$ and $f'(1) = 6$, the tangent line has equation $y - 4 = 6(x - 1)$.

(d) For the tangent line, $y = 6(x - 1) + 4$, so $f(1.1) \approx 6(0.1) + 4 = 4.6$.

6. (15 points) True or False/fill-in. Assume g is a function that is continuous and differentiable everywhere.

   (a) ____________ If $f(x)$ is a polynomial of degree 5, then f can have as many as 6 extrema.

   (b) ____________ The slope of the tangent line to the graph of $y = f(x)$ at $x = a$ is $f'(a)$.

   (c) ____________ The function $f(x) = x^2 - 4x + 3$ has a horizontal tangent line in $[1, 3]$.

   (d) ____________ If the points $(1, 3)$ and $(4, 6)$ are on the graph of g, then $g'(x)$ **must** equal 2 somewhere in the interval $(1, 3)$.

   (e) ____________ If the points $(1, 3)$ and $(4, 6)$ are on the graph of g, then $g'(x)$ **must** equal 1 somewhere in the interval $(1, 3)$.

   Solution: (a) False; it can have at most 4. (b) True; the derivative gives the slope of the tangent line. (c) True; since $f(1) = 0 = f(3)$, Rolle's Theorem guarantees some $c \in (1, 3)$ such that $f'(c) = 0$. (d) False; the AROC is 1, so there is no guarantee the derivative will equal 2. (e) True; the AROC is 1 and g is differentiable and continuous everywhere (and hence on $(1, 3)$ and $[1, 3]$, respectively), so the MVT guarantees some $c \in (1, 3)$ such that $f'(c) = 1$.

7. (10 points) Find an equation of the tangent line to $f(x) = x + x^{1/3}$ at $x = 1$.

   Solution: $f'(x) = 1 + \frac{1}{3}x^{-2/3}$, so $f'(1) = \frac{4}{3}$. Since $f(1) = 2$, the tangent line has equation $y - 2 = \frac{4}{3}(x - 1)$. (No need to solve for y.)

8. (10 points) **BONUS!!!** Use the definition of the derivative to prove that

$$((f + g)(x))' = f'(x) + g'(x).$$

   Solution: See the book.