MATH 150

Exam 3 Solutions **Friday, November 6, 2020**

Name:__________________________

Remember to **show your work**. Unsupported solutions will receive no credit.

> **The calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word.**
> **– W. F. Osgood, quoted in Bulletin American Mathematical Society**

1. (10 points) Given that $\sin(\theta) = \frac{3}{7}$, determine each quantity below **exactly** (i.e., don't use your calculator to approximate a decimal answer).

   (a) $\cos(\theta) =$______________

   (b) $\sin(2\theta) =$______________

   (c) $\tan(\theta) =$______________

   Solution: Create a right triangle with another angle labeled θ. Then we have the side opposite θ of length 3 and the hypotenuse of length 7 so that $\sin(\theta) = \frac{3}{7}$ as is given. The Pythagorean Theorem tells us the remaining leg of length x, say, must satisfy $x^2+3^2 = 7^2$, so $x^2 = 49-9 = 40$. Thus $x = \sqrt{40}$. Now we can determine $\cos\theta = \frac{\sqrt{40}}{7}$, $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{3}{7}\cdot\frac{\sqrt{40}}{7}$, and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{7}}{\frac{\sqrt{40}}{7}} = \frac{3}{\sqrt{40}}$.

2. (30 points) Differentiate each function. **You do not need to simplify your answers.**

   (a) $f(x) = e^{\cos(x)}$
   Solution: $f'(x) = e^{\cos(x)}(-\sin(x))$ using either the Chain Rule or our Exponential Derivative Rule $\left(\frac{d}{dx}(e^{g(x)}) = g'(x)e^{g(x)}\right)$.

   (b) $f(x) = x^2\sin(x)$
   Solution: Use the Product Rule to obtain $f'(x) = 2x\sin(x) + \cos(x)\cdot x^2$.

   (c) $f(x) = \arctan(e^{6x})$
   Solution: Use the Chain Rule twice or the Chain Rule in combination with the Exponential Rule: $\frac{1}{1+(e^{6x})^2}\cdot(6e^{6x})$.

   (d) $f(x) = \tan(4x^2)$
   Solution: Chain Rule: $f'(x) = \sec^2(4x^2)\cdot 8x$.

   (e) $f(x) = (\ln(x+2) + \sec(5x^2))^4$
   Solution: Chain Rule a few times: $f'(x) = 4(\ln(x+2)+\sec(5x^2))^3\left(\frac{1}{x+2} + \sec(5x^2)\tan(5x^2)\cdot 10x\right)$.

   (f) $f(x) = x^x$
   Solution: $x^x = e^{x\ln(x)}$, so we get $f'(x) = \left(\ln(x) + \frac{1}{x}\cdot x\right)e^{x\ln(x)}$ using our Exponential Rule and using the Product Rule for the derivative of the exponent.

3. (10 points) Saruman is able to grow his Uruk-hai army at a rate of 18% every 14 days. How long does it take for Saruman to double the size of his army?

   Solution 1: Let $A(t)$ represent the size of Saruman's army after t days. Then 18% growth in 14 days corresponds to a model of the form $A(t) = A_0(1.18)^{t/14}$, where A_0 is the initial size of his army. To reach $2A_0$, then (for doubling), we need to solve $A_0(1.18)^{t/14} = 2A_0$.

$$
\begin{aligned}
A_0(1.18)^{t/14} &= 2A_0 \\
(1.18)^{t/14} &= 2 \\
\ln(1.18^{t/14}) &= \ln 2 \\
\frac{t}{14}\ln 1.18 &= \ln 2 \\
t &= \frac{14\ln 2}{\ln 1.18} \\
&\approx 58.63
\end{aligned}
$$

days.

Solution 2: With the same notation as above, $A(t) = A_0 e^{kt}$, where k is an unknown parameter we need to find. We are given that $A(14) = 1.18A_0$ (from 18% growth), and our model says $A(14) = A_0 e^{k(14)}$. We thus need to solve $1.18A_0 = A_0 e^{14k}$ for k:

$$
\begin{aligned}
1.18A_0 &= A_0 e^{14k} \\
1.18 &= e^{14k} \\
\ln(1.18) &= \ln(e^{14k}) \\
\ln(1.18) &= 14k \\
k &= \frac{\ln(1.18)}{14}.
\end{aligned}
$$

With k in hand, we now have a complete model: $A(t) = A_0 e^{t\ln(1.18)/14}$. We just need to solve for the doubling time:

$$
\begin{aligned}
2A_0 &= A_0 e^{t\ln(1.18)/14} \\
2 &= e^{t\ln(1.18)/14} \\
\ln(2) &= \frac{t\ln(1.18)}{14} \\
t &= \frac{14\ln(2)}{\ln(1.18)},
\end{aligned}
$$

as above!

Note that the "solving for the doubling time" part was about the same amount of work in both solutions, but the second solution required solving for k, as well – quite a few more steps. This is why I prefer the first solution method.

4. (10 points) Grond is swung back and forth in an attempt to breach the gates of Minas Tirith. At time t seconds, Grond's snout is $x(t) = 12 + 12\cos\left(\frac{\pi t}{6}\right)$ meters from the gates of Minas Tirith. Find the speed of Grond's snout at time $t = 1$s.

   Solution: We need $x'(1)$. Differentiating gives us $x'(t) = -12\sin\left(\frac{\pi t}{6}\right)\cdot\frac{\pi}{6}$, so $x'(1) = -12\sin\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{6} = -12(1/2)(\pi/6) = -\pi$.

5. (20 points) True or False/fill-in.

   (a) **True** The domain of $\cos(x)$ is $(-\infty, \infty)$.

   (b) **False** The domain of $\tan(x)$ is $(-\infty, \infty)$. **The tangent function has holes in its domain at odd multiples of $\pi/2$.**

   (c) **False** $\sqrt{x^2+4} = x + 2$ for all $x \geq 0$.

(d) **True** $\ln(ab) = \ln(a) + \ln(b)$ for all $a, b > 0$.

(e) **False** $\ln(a + b) = \ln(a) + \ln(b)$ for all $a, b > 0$.

(f) A sample of an element with a half-life of 12s has **25**% remaining after 24s. **(That's two half lives, so half of half remains.)**

(g) $\lim_{x\to\infty} \frac{e^{5x}}{3 + 2e^{5x}}$ = **1/2** **(Use L'Hôpital's Rule .)**

(h) $\lim_{x\to\infty} \frac{x^{2020}}{1.00001^x}$ = 0 **(Use L'Hôpital's Rule over and over and over and over and... or note that exponentials dominate polynomials.)**

(i) $\lim_{x\to\infty} \frac{4x^6 - 3x^4 + 2x^2 - 8x + 9}{7x^6 - 2x^5 + 3x^3 + 2x - 4}$ = **4/7** **(ROTLC)**

(j) $\lim_{x\to\infty} e^{-3x}$ = **0**

6. (10 points) A surveyor measures an angle of 42° between the ground and the top of a building. If she is 50 feet from the building, how tall is the building?

   Solution: The ground, building, and line of site from the surveyor to the top of the building form a right triangle in which the side adjacent to the given angle is 50 feet. Thus, $\tan(42°) = \frac{h}{50}$, where h is the height of the building. Therefore $h = 50\tan(42°) \approx 45$ feet. (Note: if you use your calculator to find the cosine, make sure it's in degree mode.)

7. (10 points) Find an equation of the tangent line to $f(x) = \sin(4x)$ at $x = \frac{\pi}{6}$.

   Solution: As always, we need a point and the slope. The point is at $x = \frac{\pi}{6}$, which gives $y = f(\pi/6) = \sin(4\pi/6) = \frac{\sqrt{3}}{2}$. The slope comes from (all together now!) the derivative: $f'(x) = \cos(4x) \cdot 4$, so $f'(\pi/6) = 4\cos(4\pi/3) = 4(-1/2) = -2$. Thus the tangent line has equation $y - \frac{\sqrt{3}}{2} = -2\left(x - \frac{\pi}{6}\right)$.

8. **BONUS!!!** (10 points) Express $\cos(3\theta)$ in terms of $\cos(\theta)$.

   Solution: Just a hint since it's a bonus problem: $\cos(3\theta) = \cos(2\theta + \theta)$.