MTH 249

Solutions to Exam 1 **Tuesday, September 21, 2004**

Name:________________________

Remember to **show your work**. Unsupported solutions will receive **no credit**.

1. (24 points) Let $u =< 3,1,2 >, v =< 1,5,-2 >, w =< 4,-1,1 >, x =< 7,2 >$, and $y =< -2,1 >$. Compute any quantity that is meaningful. If it is not meaningful, explain why not.

   (a) $u + v$
   Solution: $u + v =< 3,1,2 > + < 1,5,-2 >=< 3+1, 1+5, 2+(-2) >=< 4,6,0 >$.

   (b) $v \cdot w$
   Solution: $v \cdot w =< 1,5,-2 > \cdot < 4,-1,1 >= (1)(4) + (5)(-1) + (-2)(1) = -3$.

   (c) $3x$
   Solution: $3x = 3 < 7,2 >=< 3(7), 3(2) >=< 21,6 >$.

   (d) $y - 2u$
   Solution: This is not defined; the vectors are different dimensions.

   (e) $v \times y$
   Solution: This is not defined; a cross product requires two three-dimensional vectors.

   (f) $w \times u$
   Solution: $w \times u =< 4,-1,1 > \times < 3,1,2 >=< (-1)(2) - 1(1), -(4(2) - 1(3)), 4(1) - 3(-1) >=< -3,-5,7 >$.

   (g) $(u \cdot v) \cdot w$
   Solution: This is not defined; $u \cdot v$ is a scalar, and a scalar cannot be "dotted" with a vector.

   (h) $u \times (w \cdot v)$
   Solution: This is not defined; $w \cdot v$ is a scalar, and a scalar cannot be "crossed" with a vector.

2. (6 points) Find a unit vector in the direction of $u =< 3,-4,3 >$.

   Solution: $\dfrac{< 3,-4,3 >}{\sqrt{3^2 + (-4)^2 + 3^2}} = \dfrac{1}{\sqrt{34}} < 3,-4,3 >$ is a unit vector in that direction.

3. (10 points)

   (a) (4 points) Find an equation of the sphere centered at $(3,1,-1)$ with radius 4.
   Solution: $(x-3)^2 + (y-1)^2 + (z+1)^2 = 4^2$.

   (b) (6 points) Show that the equation $x^2 + y^2 + z^2 = 4x - 2y + 6z + 1$ defines a sphere. What are the center and radius of the sphere?
   Solution: Complete the square:

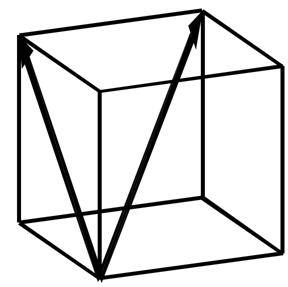
$$
\begin{aligned}
x^2 + y^2 + z^2 &= 4x - 2y + 6z + 1 \\
(x^2 - 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) &= 1 + 4 + 1 + 9 \\
(x-2)^2 + (y+1)^2 + (z-3)^2 &= 15.
\end{aligned}
$$

   The center is thus $(2,-1,3)$ and the radius is $\sqrt{15}$.

4. (8 points) Recall that the torque τ about point P generated by a force F applied at a point Q is given by $\tau = \vec{PQ} \times F$. Find the torque generated by a force $F =< 3,1,6 >$ if $\vec{PQ} =< 2,6,3 >$.

   Solution: $< 2,6,3 > \times < 3,1,6 >=< 36 - 3, -(12 - 9), 2 - 18 >=< 33,-3,-16 >$.

5. (10 points) What is the angle between the main diagonal of a cube and a face diagonal? (See the figure.)



Solution: Taking one of the bottom corners to be $(0, 0, 0)$ and the other two corners as $(1, 0, 1)$ and $(1, 1, 1)$, the vectors describing these diagonals are $< 1, 0, 1 >$ and $< 1, 1, 1 >$. The angle between these is $\arccos \dfrac{< 1, 0, 1 > \cdot < 1, 1, 1 >}{\sqrt{2}\sqrt{3}} = \arccos \dfrac{2}{\sqrt{6}} \approx 35.26$ degrees.

6. (10 points) $\vec{PQ}$, $\vec{PR}$, and $\vec{PS}$ are adjacent edges of a parallelepiped. If the the points are $P(-2, 3, 1)$, $Q(2, 3, 6)$, $R(1, -1, 0)$, and $S(1, 3, 1)$, what is the volume of the parallelepiped?

   Solution: $\vec{PQ} =< 4, 0, 5 >, \vec{PR} =< 3, -4, -1 >$, and $\vec{PS} =< 3, 0, 0 >$. The volume is thus $| < 4, 0, 5 > \cdot (< 3, -4, -1 > \times < 3, 0, 0 >)| = | < 4, 0, 5 > \cdot < 0, -3, 12 > | = 60$.

7. (12 points) What are the scalar and vector projections of u onto v if $u =< 4, 1, -5 >$ and $v =< 2, 6, 1 >$?

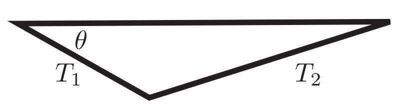
   Solution: Scalar: $\dfrac{u \cdot v}{|v|} = \dfrac{9}{\sqrt{41}}$.

8. (6 points) Draw vectors u and v so that $u \times v$ is directed **into** the page.

   Solution: If u points to 1 on a clock face and v points to 3, then $u \times v$ will go into the page.

9. (14 points) A tightrope walker walks one-third of the way across a 60-foot tightrope and pauses. She weighs 135 pounds and deflects the tightrope down by 1.5 feet at this point. What is the magnitude of the tension in each end of the tightrope?

   Solution: The downward deflection indicates the angles: $\tan\theta_1 = \dfrac{1.5}{20}$ and $\tan\theta_2 = \dfrac{1.5}{40}$, so $\theta_1 \approx 4.28915$ degrees and $\theta_2 \approx 2.14759$ degrees.



Now because the acrobat is in equilibrium, the forces balance. The vertical forces are $T_1 \sin\theta_1 + T_2 \sin\theta_2 = 135$. The horizontal forces are $T_1 \cos\theta_1 = T_2 \cos\theta_2$. Solving for T_1 and substituting gives $T_2 = \dfrac{135\cos\theta_1}{\sin(\theta_1 + \theta_2)} \approx 1200.84$ pounds. Thus $T_1 \approx 1203.37$ pounds.

10. **Bonus!** (5 points) Show that if u and v are parallel, then the projection of u onto v is $\pm u$.