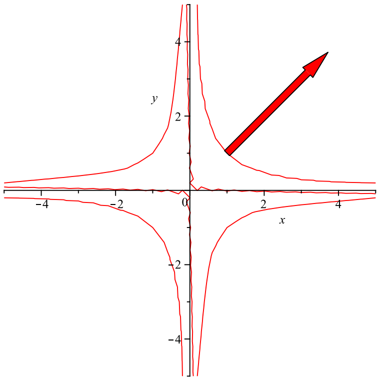
Math 249 Exam III

Tuesday, November 16, 2004

Remember to **show all work**. Unsupported solutions will receive **no credit**.

1. (20 points) Consider $f(x, y) = e^{xy}$.
   (a) Find $\nabla f(x, y)$.
   Solution: $\nabla f(x,y) = < f_x(x,y), f_y(x,y) > = < ye^{xy}, xe^{xy} >$.
   (b) Sketch the level curves for $z = e^{-1}, 1, e$. Sketch as accurately as possible and label each.
   Solution: (for both (b) and (c))



   (c) Draw the gradient at $(1, 1)$ on your level curves.
   (d) What does $\nabla f(1, 1)$ mean? (I'm not looking for a name; I'm looking for an interpretation!)
   Solution: This is the direction of greatest increase on the graph of f at the point $(1, 1)$.
   (e) Find the directional derivative of f in the direction of $< -3, 4 >$ at $(1, 1)$. How does this number compare with the length of $\nabla f(1, 1)$? Why?
   Solution: $D_u f(1,1) = < e, e > \cdot \dfrac{1}{5} < -3, 4 > = \dfrac{e}{5}$. It is smaller than the length of $\nabla f(1, 1)$ since the latter is the maximum rate of increase (for any direction) at $(1, 1)$.

2. (10 points) As a car climbs a mountain, a scientist is measuring atmospheric pressure. She determines that the pressure at (x, y) (coordinates on a map) is given by $P(x, y) = 70 + 3\sqrt{x^2 + y^2}$ kPa. If $x(t) = (10 - t)\cos(2\pi t)$ and $y(t) = (10 - t)\sin(2\pi t)$, where t is measured in hours, find $\dfrac{dP}{dt}$ at $t = 2$ hours.
   Solution:

$$\begin{aligned}\frac{dP}{dt} &= \frac{dP}{dx}\frac{dx}{dt} + \frac{dP}{dy}\frac{dy}{dt} \\ &= \frac{3x}{\sqrt{x^2+y^2}}(-\cos(2\pi t) - 2\pi(10-t)\sin(2\pi t) + \frac{3y}{\sqrt{x^2+y^2}}(-\sin(2\pi t) - 2\pi(10-t)\cos(2\pi t).\end{aligned}$$

   At $t = 2$, we have $x = 8$ and $y = 0$, so $\dfrac{dP}{dt} = -3$.

3. (10 points) A toy manufacturer has a mold to create a plastic "Rapunzel's Tower," which is a cone placed on top of a cylinder. The inner radius of each is 5cm, the inside heights of the cylinder and cone are each 15 cm. The cylinder has a bottom. If the plastic is 0.15 cm thick, use differentials to estimate the amount of plastic in each Rapunzel's Tower.
   Solution: $V = \dfrac{1}{3}\pi r^2 h + \pi r^2 h$, so $dV = \dfrac{4}{3}(2\pi rh dr + \pi r^2 dh) = \dfrac{4}{3}(2\pi(5)(15)(0.15) + \pi(5^2)(0.15) = 35\pi \approx 110 \text{ cm}^3$.

4. (20 points) Find the critical points of $f(x, y) = 2x^2 - 4xy + y^3 + 2$ and classify each as a local maximum, local minimum, or neither.
   Solution: $f_x = 4x - 4y$, $f_y = -4x + 3y^2$. From $f_x = 0$, we get $y = x$, so $f_y = 0$ implies $3y^2 = 4y$. Thus $y = 0$ or $y = 4/3$. The critical points are $(0, 0)$ and $(4/3, 4/3)$. (Recall we have $y = x$.)
   Now $f_{xx} = 4$, $f_{xy} = -4$, and $f_{yy} = 6y$, so $D(x, y) = 24y - 16$. $D(0, 0) = -16 <$, so $(0, 0)$ is a saddle point. $D(4/3, 4/3) = 16 > 0$. $f_{xx}(4/3, 4/3) = 4 > 0$, so we have a local minimum at $(4/3, 4/3)$.

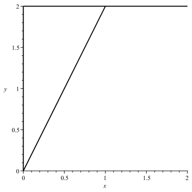
5. (10 points) Compute $\iint_D xe^{xy}dA$ if $D = [0, 1] \times [-1, 1]$.
   Solution:

$$\begin{aligned}\iint_D xe^{xy}dA &= \int_0^1 \int_{-1}^1 xe^{xy}dydx \\ &= \int_0^1 (e^x - e^{-x})dx \\ &= e^x + e^{-x}\Big|_0^1 \\ &= e + e^{-1} - 2.\end{aligned}$$

6. (10 points) Compute $\iint_D \frac{1}{\sqrt{4 - x^2 - y^2}}dA$, where $D = \{(x, y)|x \geq 0, y \geq 0, \text{ and } 1 \leq x^2 + y^2 \leq 4\}$.
   Solution:

$$\begin{aligned}\iint_D \frac{1}{\sqrt{4 - x^2 - y^2}}dA &= \int_0^{\pi/2} \int_1^2 \frac{r}{\sqrt{4 - r^2}}drd\theta \\ &= \frac{\pi}{2}(-\sqrt{4 - r^2})\Big|_1^2 \\ &= \frac{\pi\sqrt{3}}{2}.\end{aligned}$$

7. (10 points) Compute $\int_0^1 \int_{2x}^2 e^{-y^2}dydx$ by reversing the order of integration. Sketch the region of integration.
   Solution: The region of integration is the triangle shown below.



We get

$$\begin{aligned}\int_0^1 \int_{2x}^2 e^{-y^2}dydx &= \int_0^2 \int_0^{y/2} e^{-y^2}dxdy \\ &= \int_0^2 \frac{y}{2}e^{-y^2}dy \\ &= -\frac{1}{4}e^{-y^2}\Big|_0^2 \\ &= \frac{1 - e^{-4}}{4}.\end{aligned}$$

8. (10 points) True or False.

   (a) <u>TRUE</u> If L is the linearization of f at (x_0, y_0), then the graph of L is the tangent plane to the graph of f at (x_0, y_0).

   (b) <u>FALSE</u> For any function f of two variables, $f_{xy} = f_{yx}$.

   (c) <u>FALSE</u> If f is continuous on the bounded set D, then f has an absolute maximum and minimum on D.

   (d) <u>TRUE</u> If f is continuous on $[a, b] \times [c, d]$, then $\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$.

   (e) <u>FALSE</u> If f is a function of two variables, then $\nabla f(x_0, y_0)$ is perpendicular the tangent plane to the graph of f at (x_0, y_0).