Math 249 Final Exam

Monday, December 13, 2004

Remember to **show all work**. Unsupported solutions will receive **no credit**.

You *should* know these by now, but here they are:

$$\cos 2t = \cos^2 t - \sin^2 t, \sin 2t = 2\sin t \cos t, \cos^2 t = \frac{1+\cos 2t}{2}, \sin^2 t = \frac{1-\cos 2t}{2}.$$

1. (4 points) Find the angle between $< 1, 2, -3 >$ and $< -1, 0, 4 >$.

2. (10 points) Consider the integral $\int_C 3x^2 dx + 2xy dy$, where C is the unit circle $x^2 + y^2 = 1$.

   (a) (4 points) Compute the integral by parameterizing the unit circle.

   (b) (4 points) Compute the integral by using Green's Theorem.

   (c) (2 points) Which method did you find easier, and why?

3. (10 points)

   (a) (2 points) Show that $F =< 3x^2y, x^3 + 1 >$ is a conservative vector field.

   (b) (4 points) Find a potential function for F.

   (c) (4 points) Compute $\int_C F \cdot dr$, where C is the path parameterized by $r(t) =< t^3e^t + 1, t^2 - 4t >$ on $[0, 1]$.

4. (8 points)

   (a) (3 points) Parameterize the line segment C from $(3, 0, 1)$ to $(-2, 4, 5)$.

   (b) (5 points) Set up completely $\int_C (x^2y - 2z)ds$, but do not evaluate it.

5. (12 points) Consider the function $f(x, y) = \frac{x^2}{4} + y^2$.

   (a) Draw the level curves for $z = 0, 1, 2, 3, 4$ as accurately as possible, including scale.

   (b) Draw the gradient of f at the point $(-0.5, 1)$ on your set of level curves.

   (c) Sketch the graph of f. Indicate where your level curves from (a) are on the graph.

   (d) Are your graphs of f and your gradient/level curves consistent? That is, does the gradient seem to point the way you think it should? Explain.

6. (4 points) Convert the integral to cylindrical coordinates, but do not evaluate it.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xy\sqrt{x^2+y^2}\,dzdydx$$

7. (8 points) Find the local minima, maxima, and saddle points on the graph of

$$z = 3x^2 + 12x + 8y^3 - 12y^2 + 7.$$

8. (6) Calculate each limit or show that it does not exist:

   (a) $\lim_{(x,y)\to(1,2)} \frac{x^2 + y^2 - 5}{2x^2 - y^2 + 3}$

   (b) $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$

9. (12) Let $f(x,y) = e^{x-y}$.

   (a) What is the direction of greatest increase of f at the point $(2,1,e)$?

   (b) What is the directional derivative of f in the direction $< 1, 1 >$ at the point $(2,1,e)$?

   (c) Find an equation of the tangent plane to the graph of f at the point $(2,1,e)$.

10. (5 points) Let $f(x,y,z) = x^2 z$ and let S be the surface given by $z = 20 - 4x^2 - 4y^2$ above the plane $z = 4$. Set up the integral $\iint_S f(x,y,z)dS$, but do not evaluate it. (Get it to the stage where the next step is evaluation.)

11. (11 points) Let $\overline{F} =< e^x, ye^x, 4z >$.

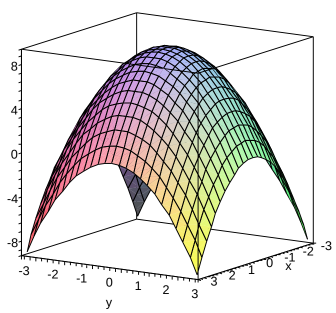
   (a) (3 points) Compute curl$\overline{F}$.

   (b) (3 points) Compute div $\overline{F}$.

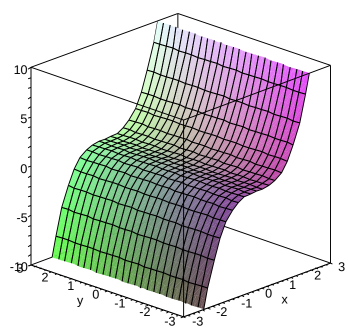
   (c) (5 points) Compute $\int_C F \cdot d\vec{r}$, where C is the boundary of the surface given by $z = e^{-x} + e^{-y}$ above the rectangle $[0,1] \times [0,2]$ with upward orientation.

12. (5 points) Compute the work done by the force $\overline{F} =< xy^2, 3x + 1 >$ along the path $\overline{r}(t) =< 4t, t^3 >$ from $t = 0$ to $t = 2$.

13. (5 points) Match each graph $z = f(x,y)$ with its gradient vector field.



(a)



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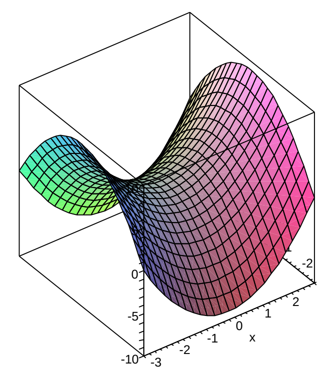
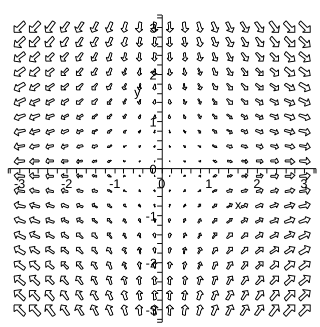
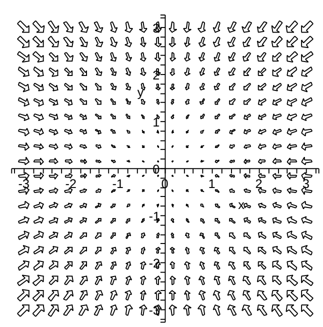


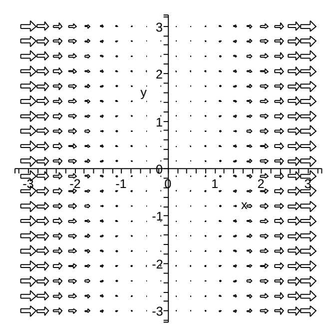
(c)



I



II



III

14. (5 points) **BONUS!** Integrate $\int\int_R \frac{2y + x}{y - 2x} dA$, where R is the trapezoid with vertices $(-1,0), (-2,0), (0,4), (0,2)$.