

# Math 249 Final Exam

Monday, December 13, 2004

Remember to **show all work**. Unsupported solutions will receive **no credit**.

You *should* know these by now, but here they are:

$$\cos 2t = \cos^2 t - \sin^2 t, \sin 2t = 2 \sin t \cos t, \cos^2 t = \frac{1 + \cos 2t}{2}, \sin^2 t = \frac{1 - \cos 2t}{2}.$$

1. (4 points) Find the angle between  $\langle 1, 2, -3 \rangle$  and  $\langle -1, 0, 4 \rangle$ .

**Solution:**  $\cos \theta = \frac{\langle 1, 2, -3 \rangle \cdot \langle -1, 0, 4 \rangle}{\sqrt{1^2 + 2^2 + (-3)^2} \sqrt{(-1)^2 + 0^2 + 4^2}} = \frac{-13}{\sqrt{14}\sqrt{17}}$ . Thus  $\theta = \arccos \frac{-13}{\sqrt{14}\sqrt{17}}$ .

2. (10 points) Consider the integral  $\int_C 3x^2 dx + 2xy dy$ , where  $C$  is the unit circle  $x^2 + y^2 = 1$ .

- (a) (4 points) Compute the integral by parameterizing the unit circle.

**Solution:** Let  $x = \cos \theta, y = \sin \theta$  on  $[0, 2\pi]$ . Then  $dx = -\sin \theta d\theta$  and  $dy = \cos \theta d\theta$ , and we get

$$\begin{aligned} \int_0^{2\pi} (3 \cos^2 \theta (-\sin \theta) + 2 \cos \theta (\cos \theta)) d\theta &= \int_0^{2\pi} (-3 \cos^2 \theta \sin \theta + (1 + \cos(2\theta))) d\theta \\ &= 2\pi. \end{aligned}$$

- (b) (4 points) Compute the integral by using Green's Theorem.

**Solution:**  $\int_C 3x^2 dx + 2xy dy = \iint_D (2 - 0) dA = 2\pi(1)^2 = 2\pi$ . (Note that  $\iint_D 2 dA = 2A$ , where  $A$  is the area of  $D$ .)

- (c) (2 points) Which method did you find easier, and why?

**Solution:** I found Green's Theorem easier – much less work to do, and no parametrization necessary!

3. (10 points)

- (a) (2 points) Show that  $F = \langle 3x^2 y, x^3 + 1 \rangle$  is a conservative vector field.

**Solution:** The domain of  $F$  is  $\mathbb{R}^2$ , which is open and simply connected.  $Q_x = 3x^2 = P_y$ , so  $F$  is conservative.

- (b) (4 points) Find a potential function for  $F$ .

**Solution:**  $f(x, y) = \int 3x^2 y dx = x^3 y + g(y)$ . Also,  $f(x, y) = \int (x^3 + 1) dy = x^3 y + y + h(x)$ . Thus, let  $f(x, y) = x^3 y + y$ .

- (c) (4 points) Compute  $\int_C F \cdot dr$ , where  $C$  is the path parameterized by  $r(t) = \langle t^3 e^t + 1, t^2 - 4t \rangle$  on  $[0, 1]$ .

**Solution:** Since  $F$  is conservative, we can apply the Fundamental Theorem for Line Integrals:  $r(0) = \langle 1, 0 \rangle$  and  $r(1) = \langle e + 1, -3 \rangle$ , so  $\int_C F \cdot dr = f(e + 1, -3) - f(1, 0) = 3(e + 1)^3 + 3$ .

4. (8 points)

- (a) (3 points) Parameterize the line segment  $C$  from  $(3, 0, 1)$  to  $(-2, 4, 5)$ .

**Solution:** Let  $x = 3 - 5t, y = 0 + 4t, z = 1 + 4t$  on  $[0, 1]$ .

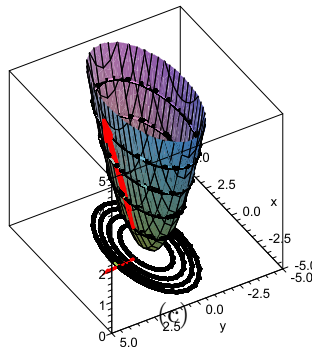
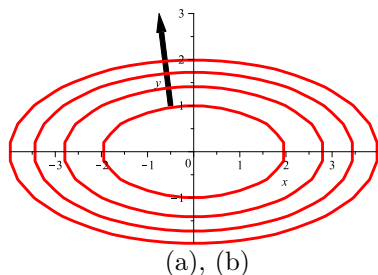
- (b) (5 points) Set up completely  $\int_C (x^2 y - 2z) ds$ , but do not evaluate it.

**Solution:**  $\int_0^1 ((3 - 5t)^2 (4t) - 2(1 + 4t)) \sqrt{(-5)^2 + 4^2 + 4^2} dt$ .

5. (12 points) Consider the function  $f(x, y) = \frac{x^2}{4} + y^2$ .

- (a) Draw the level curves for  $z = 0, 1, 2, 3, 4$  as accurately as possible, including scale.

**Solution:** This includes the solutions to (b) and (c), as well.



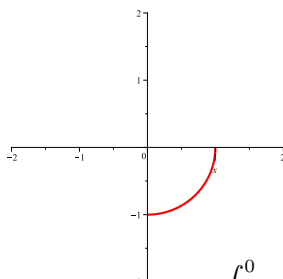
- (a), (b)
- (b) Draw the gradient of  $f$  at the point  $(-0.5, 1)$  on your set of level curves.
- (c) Sketch the graph of  $f$ . Indicate where your level curves from (a) are on the graph.
- (d) Are your graphs of  $f$  and your gradient/level curves consistent? That is, does the gradient seem to point the way you think it should? Explain.

**Solution:** They do appear to be consistent. The gradient points out away from the origin and seems to at least be close to the direction of greatest increase.

6. (4 points) Convert the integral to cylindrical coordinates, but do not evaluate it.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xy\sqrt{x^2+y^2} dz dy dx$$

**Solution:** Note that  $z$  goes from  $x^2 + y^2 = r^2$  to  $\sqrt{x^2 + y^2} = r$ . Now  $y$  goes from  $-\sqrt{1-x^2}$  to 0, which is the bottom half of the circle  $x^2 + y^2 = 1$ . Since  $x$  goes from 0 to 1, this is the bottom right quarter-circle.



Thus, we get  $\int_{-\pi/2}^0 \int_0^1 \int_{r^2}^r (r \cos \theta)(r \sin \theta)(r) r dz dr d\theta = \int_{-\pi/2}^0 \int_0^1 \int_{r^2}^r r^4 \cos \theta \sin \theta dz dr d\theta$ .

7. (8 points) Find the local minima, maxima, and saddle points on the graph of

$$z = 3x^2 + 12x + 8y^3 - 12y^2 + 7.$$

**Solution:**  $f_x = 6x + 12$ ,  $f_y = 24y^2 - 24y$ . Thus we must have  $x = -2$  and  $y = 0$  or  $1$ . This gives us two critical points:  $(-2, 0)$  and  $(-2, 1)$ .

$f_{xx} = 6$ ,  $f_{xy} = 0$ , and  $f_{yy} = 48y - 24$ . Thus  $D = f_{xx}f_{yy} - (f_{xy})^2 = 144(2y - 1)$ . At  $y = 0$ , this is negative, so we have a saddle point. At  $y = 1$ , this is positive and  $f_{xx}(-2, 0) = 6 > 0$ , so we have a local minimum.

8. (6) Calculate each limit or show that it does not exist:

(a)  $\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + y^2 - 5}{2x^2 - y^2 + 3}$

**Solution:** This is continuous at  $(1, 2)$ , so we get  $\frac{1^2 + 2^2 - 5}{2(1)^2 - 2^2 + 3} = \frac{0}{1} = 0$ .

(b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^3 + y^3}$

**Solution:** Along  $y = 0$ , we get  $\lim_{x \rightarrow 0} \frac{x^3}{x^3} = 1$ . Along  $x = 0$ , we get  $\lim_{y \rightarrow 0} \frac{-y^3}{y^3} = -1$ . Since these limits do not agree, the limit does not exist.

9. (12) Let  $f(x, y) = e^{x-y}$ .

(a) What is the direction of greatest increase of  $f$  at the point  $(2, 1, e)$ ?

**Solution:**  $\nabla f = \langle e^{x-y}, -e^{x-y} \rangle$ , so  $\nabla f(2, 1) = \langle e, -e \rangle$ .

(b) What is the directional derivative of  $f$  in the direction  $\langle 1, 1 \rangle$  at the point  $(2, 1, e)$ ?

**Solution:** Let  $u = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$  (a unit vector in the given direction). Then  $D_u f(2, 1) = \langle e, -e \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle = 0$ .

(c) Find an equation of the tangent plane to the graph of  $f$  at the point  $(2, 1, e)$ .

**Solution:**  $z - e = e(x - 2) - e(y - 1)$ . (The coefficients of  $x - 2$  and  $y - 1$  are the partials, which appear in  $\nabla f$ .)

10. (5 points) Let  $f(x, y, z) = x^2 z$  and let  $S$  be the surface given by  $z = 20 - 4x^2 - 4y^2$  above the plane  $z = 4$ . Set up the integral  $\iint_S f(x, y, z) dS$ , but do not evaluate it. (Get it to the stage where the next step is evaluation.)

**Solution:**  $S$  and the given plane intersect for  $20 - 4x^2 - 4y^2 = 4$ , or  $x^2 + y^2 = 4$ . This suggests that I should use cylindrical coordinates for this surface integral, which means  $z = 20 - r^2$  on  $S$ . Also,  $dS = \sqrt{(-8x)^2 + (-8y)^2 + 1} dA$ .

$$\text{I get } \iint_S f(x, y, z) dS = \int_0^{2\pi} \int_0^2 r^2 \cos^2 \theta (20 - 4r^2) \sqrt{64r^2 + 1} r dr d\theta = \int_0^{2\pi} \int_0^2 r^3 \cos^2 \theta (20 - 4r^2) \sqrt{64r^2 + 1} dr d\theta.$$

11. (11 points) Let  $\vec{F} = \langle e^x, ye^x, 4z \rangle$ .

(a) (3 points) Compute  $\text{curl} \vec{F}$ .

**Solution:**  $\text{curl} \vec{F} = \nabla \times \vec{F} = \langle 0, 0, ye^x \rangle$ .

(b) (3 points) Compute  $\text{div} \vec{F}$ .

**Solution:**  $\text{div} \vec{F} = \nabla \cdot \vec{F} = e^x + e^x + 4 = 2e^x + 4$ .

(c) (5 points) Compute  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the boundary of the surface given by  $z = e^{-x} + e^{-y}$  above the rectangle  $[0, 1] \times [0, 2]$  with upward orientation.

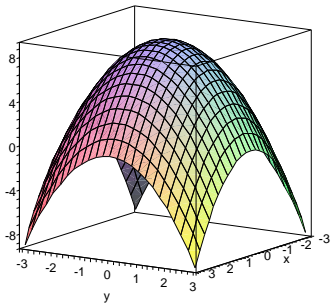
**Solution:** We may apply Stokes' Theorem. The surface is described by the function  $z = e^{-x} + e^{-y}$ , so we get

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl} \vec{F} \cdot d\vec{S} \\ &= \int_0^1 \int_0^2 (-0(-e^{-x}) - 0(-e^{-y}) + ye^x) dy dx \\ &= \left( \frac{1}{2} y^2 \Big|_0^2 \right) \left( e^x \Big|_0^1 \right) \\ &= 2(e - 1). \end{aligned}$$

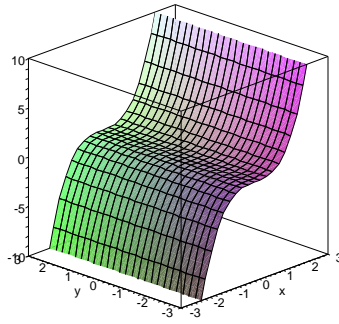
12. (5 points) Compute the work done by the force  $\vec{F} = \langle xy^2, 3x + 1 \rangle$  along the path  $\vec{r}(t) = \langle 4t, t^3 \rangle$  from  $t = 0$  to  $t = 2$ .

**Solution:** Since  $\vec{F}$  is not conservative, we must actually use the parametrization: the work done is  $\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \langle 4t(t^3)^2, 3(4t) + 1 \rangle \cdot \langle 4, 3t^2 \rangle dt = \int_0^2 (16t^7 + 36t^3 + 3t^2) dt = 664$ .

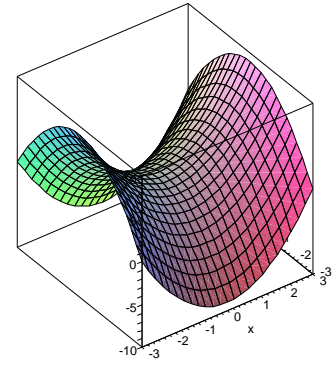
13. (5 points) Match each graph  $z = f(x, y)$  with its gradient vector field.



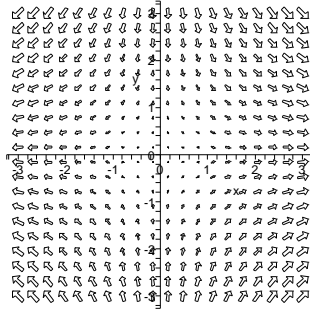
(a)



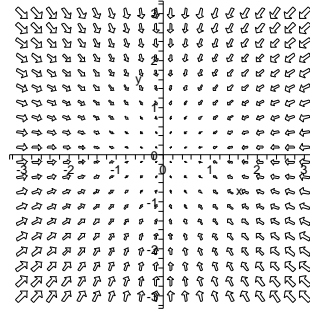
(b)



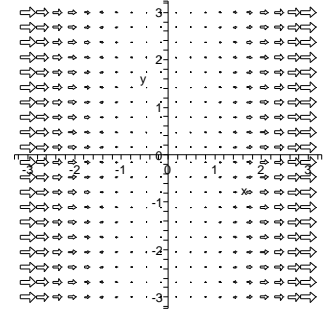
(c)



I



II



III

**Solution:** We match (b) with III since the gradient at each point is in the same direction. Also, (a) is matched with II since the direction of greatest increase at any point on (a) is toward the origin. Finally, this leaves (c) to be matched with I, which makes sense since going up along the  $y$ -axis requires moving toward the origin and going up along the  $x$ -axis requires moving away from the origin.

14. (5 points) **BONUS!** Integrate  $\iint_R \frac{2y+x}{y-2x} dA$ , where  $R$  is the trapezoid with vertices  $(-1, 0)$ ,  $(-2, 0)$ ,  $(0, 4)$ ,  $(0, 2)$ .