MATH 249

Exam 2 **Wednesday, October 16, 2013**

Name:________________________

Remember to **show your work**. Unsupported solutions will receive **no credit**.

Please leave the upper left-hand corner of each page blank so you do not staple over your work. Please write on only one side of the paper. You may use Maple as needed, but be sure to tell me when you do.

1. (10 points) Let $r(t) =< 8\cos t, 15\cos t, 17\sin t >$. Compute $T(t)$ and the curvature of r.

   Solution: $\vec{r}\,'(t) =< -8\sin t, -15\sin t, 17\cos t >$, so $|\vec{r}\,'(t)| = \sqrt{8^2\sin^2(t) + 15^2\sin^2(t) + 17^2\cos^2(t)} = \sqrt{17^2\sin^2 t + 17^2\cos^2 t} = 17$. Thus $\vec{T} = \frac{1}{17} < -8\sin t, -15\sin t, 17\cos t >$.

   We also need $\vec{r}\,''(t) =< -8\cos t, -15\cos t, -17\sin t >$. Now

   $$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{|< 255, -136, 0 >|}{17^3} = \frac{1}{17}.$$

2. (25 points) Let $f(x, y) = x^2y + xy^2 + xy$.

   (a) (10 points) Find the linearization of f at the point $(2, 1)$.
   Solution: $f_x = 2xy + y^2 + y, f_y = x^2 + 2xy + x$. Thus $f_x(2, 1) = 6, f_y(2, 1) = 10$, and $f(2, 1) = 8$, so we get $L(x, y) = 8 + 6(x - 2) + 10(y - 1)$.

   (b) (15 points) Find all extrema and saddle points of f. A graphical argument will not suffice; I need to see your analysis. [Hint: there are four critical points.]
   Solution: Using the partial derivatives from above, use Maple to solve $f_x = 0, f_y = 0$. I get the critical points $(0, 0), (0, -1), (-1, 0)$, and $(-1/3, -1/3)$.
   Also, $f_{xx} = 2y, f_{yy} = 2x$, and $f_{xy} = 2x + 2y + 1$. Thus $D = 4xy - (2x + 2y + 1)^2$.
   $(0, 0) : D(0, 0) = -1 < 0$, so this is a saddle point.
   $(0, -1) : D(0, -1) = -1 < 0$, so this is also a saddle point. By symmetry, $(-1, 0)$ is also a saddle point.
   $(-1/3, -1/3) : D(-1/3, -1/3) = 1/3 > 0$, so this is either a max or a min. To see which, evaluate $f_{xx}(-1/3, -1/3) = -2/3 < 0$. Thus the surface is concave down traveling in the x-direction, so it is concave down in every direection. Therefore, $f(-1/3, -1/3) = 1/27$ is a local maximum.

3. (10 points) Consider $f(x, y) = x^2 - y$.

   (a) Find the directional derivative of f in the direction of $< -3, 4 >$ at $(1, 2)$.
   Solution: $\nabla f(x, y) =< 2x, -1 >$, and $\nabla f(1, 2) =< 2, -1 >$. Our $\vec{u} = \frac{1}{5} < -3, 4 >$ (unit vector!), and thus $D_u f(1, 2) =< 2, -1 > \cdot \frac{1}{5} < -3, 4 >= \frac{-10}{5} = -2$.

   (b) Determine the maximum rate of increase of f at the point $(1, 2)$.
   Solution: The maximum rate of increase is $|\nabla f(1, 2)| = |< 2, -1 >| = \sqrt{5}$.

4. (10 points) The surface S is defined by $x^2 - y^2 + 6z^2 = 14$. Find an equation of the tangent plane to S at $(3, 1, 1)$.

   Solution: The normal to this level surface of $F(x, y, z) = x^2 - y^2 + 6z^2$ is $\nabla F =< 2x, -2y, 12z >$, which is $\nabla F(3, 1, 1) =< 6, -2, 12 >$ at $(3, 1, 1)$. Thus, the tangent plane has equation $6(x - 3) - 2(y - 1) + 12(z - 1) = 0$.

5. (10 points) If $f(x, y) = e^{2x+3y}$, $x(s, t) = 4s^2t$, and $y(s, t) = 2st^2$, find $\frac{\partial f}{\partial s}$. It is not necessary to simplify nor to express your final answer in terms of s and t.

   Solution: $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2e^{2x+3y}(8st) + 3e^{2x+3y}(2t^2)$.

6. (15 points) Use **differentials** to estimate the amount of aluminum needed to make a cylindrical soda can that is 12.2 cm high and has a diameter of 6.0 cm. The aluminum is 0.05 cm thick. **(No credit for any other method!)** You do not need to simplify your answer.

   Solution: The colume of a cylinder of radius r and height h is given by $V = \pi r^2 h$. We want ΔV, the difference in volume between the inside of the can and the outside of the can. Since V is differentiable, we can approximate this with dV:

$$\begin{aligned} \Delta V &\approx dV \\ &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(3)(12.2)(0.05) + \pi(3)^2(0.1). \end{aligned}$$

   Notice that dh was twice the thickness of the metal since there is metal on both the top and bottom.

7. (10 points) True or False/fill-in.

   (a) False: u must be a uinit vector. If u is a vector, then the directional derivative of f in the direction of u is $\nabla f \cdot u$.

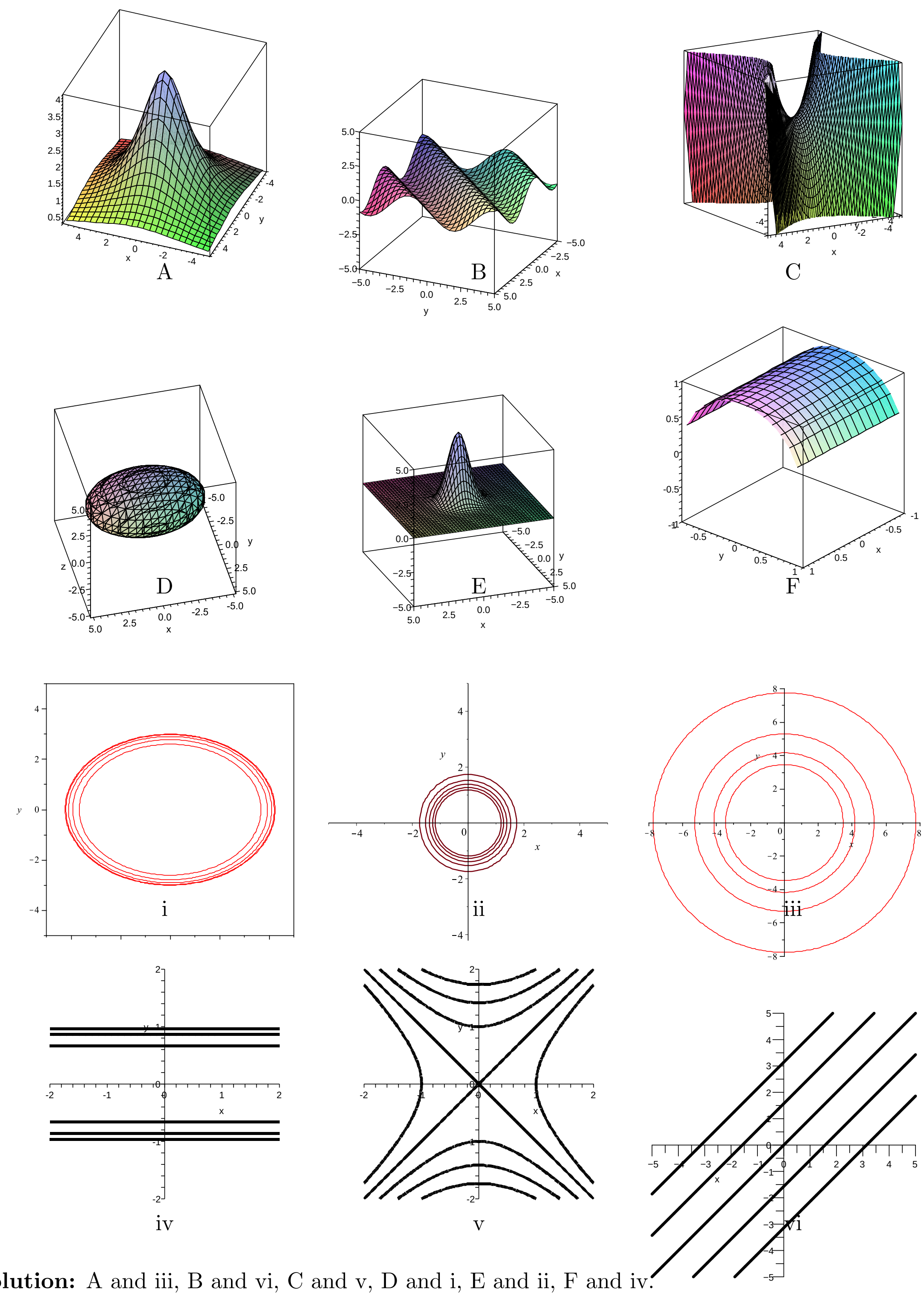
   (b) True. The graph of the linearization of a differentiable function at a point is the tangent plane to the graph of the function at that point.

   (c) False: it is perpendicular to level curves of the function. The gradient of a function is perpendicular to the graph of the function.

   (d) False: it lives in a different space. The gradient of a function is tangent to the graph of the function.

   (e) False: $\Delta z \approx dz$. If $z = f(x, y)$ is differentiable, then $\Delta z = dz$.

8. (10 points) Match the letter of each equation with a surface and a set of level curves. The level curves are drawn for at most five values of $f(x, y)$ and correspond to equal changes in z for each graph.



Solution: A and iii, B and vi, C and v, D and i, E and ii, F and iv.