# Math 249 Exam III

Wednesday, November 13, 2013

Name:____________________________________

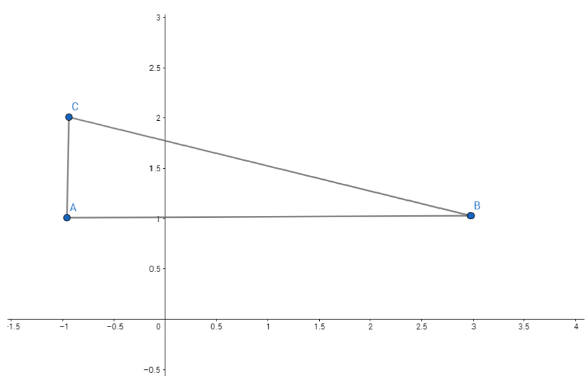
Remember to **show all work**. Unsupported solutions will receive **no credit**.

1. (50 points) Integration fundamentals. For each "set up" problem, use the coordinate system in which you would carry out the integration. (Your score depends on your choice!) For example, if spherical coordinates are more natural, set up the integral with spherical coordinates. You do not need to simplify your integrands.

   (a) Compute $\iint\limits_D 3x^2\cos(2y)dA$ if $D = [1,3]\times[0,\pi/4]$. **Integrate!** This is the one where I want to see the integration steps, so don't use Maple on this one (except to check).
   **Solution:**

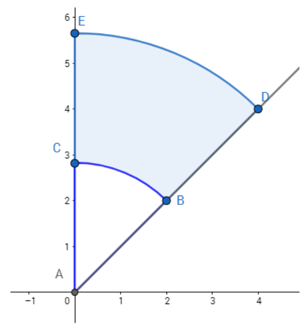
$$\begin{aligned}\iint\limits_D 3x^2\cos(2y)dA &= \int_1^3\int_0^{\pi/4} 3x^2\cos(2y)dydx\\ &= \int_1^3 3x^2\left(\frac{1}{2}\sin(2y)\right)\Bigg|_{y=0}^{y=\pi/4} dx\\ &= \int_1^3 \frac{3}{2}x^2dx\\ &= \frac{1}{2}x^3\Big|_1^3\\ &= 13.\end{aligned}$$

   (b) Set up $\iint\limits_D x^2+y^2dA$, where $D$ is the triangle with vertices $(-1,1),(3,1)$, and $(-1,2)$. **Do not integrate!**
   **Solution:** The region is graphed below. (I recommend that you **always** sketch the region if integration.)




   I see $x$ ranging from -1 to 3 and $y$ ranging from 1 up to the line, which has equation $y = -\frac{1}{4}x+\frac{7}{4}$.
   Thus, the integral becomes $\displaystyle\int_{-1}^3\int_1^{-\frac{1}{4}x+\frac{7}{4}}(x^2+y^2)dydx.$
   The integrand may have tempted you to consider polar coordinates, but the triangular region says, "Stick to Cartesian!"

   (c) Set up $\iint\limits_D \left(x\sqrt{4+x^2+y^2}\right)^3 dA$, where $D = \{(x,y)|4\le x^2+y^2\le 16, x\ge 0, \text{ and } y\ge x\}$. **Do not integrate!**
   **Solution:** Again, first sketch the region of integration:



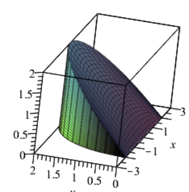

This looks like a job for polar coordinates!

$$\iint_D \left(x\sqrt{4+x^2+y^2}\right)^3 dA \int_{\pi/4}^{\pi/2}\int_2^4 ((r\cos\theta)\sqrt{4+r^2})^3 r\,dr\,d\theta$$

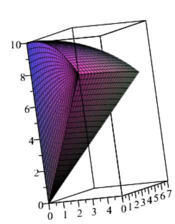
(d) Set up an integral to find the mass of a wire with density function $x^2y^2$ if the wire is in the shape of a circle of radius 4.
**Solution:** This is a path integral. On this circle, we have $x(t) = 2\cos(t), y(t) = 2\sin(t)$ on $[0, 2\pi]$, and $ds = \sqrt{4\cos^2 t + 4\sin^2 t}dt = 2dt$. We get

$$\int_C f(x,y)ds = \int_0^{2\pi} (2\cos t)^2(2\sin t)^2 2dt.$$

(e) Set up $\iiint_E e^{x^2+y^2+z^2} dV$, where $E$ is the region inside the cylinder $x^2+y^2=4$, above the plane $z=0$, and below the plane $z=y$. **Do not integrate!**
**Solution:** Use polar coordinates. Here is the region:



(f) Set up a triple integral to find the mass of the solid in the first octant bounded by the graphs of $z^2 = x^2+y^2$, $x^2+y^2+z^2 = 100, z = 0, y = 0$, and $y = x$, where the density of the solid is $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$. **Do not integrate!**
**Solution:** Here is the region:



Use spherical coordinates. The cone $z^2 = x^2+y^2$ and the sphere $x^2+y^2+z^2 = 100$ meet at

$z^2 + z^2 = 100$, or $z = \sqrt{50}$. Also, note that on the cone, $\dfrac{z^2}{x^2+y^2} = 1$; that is, $\dfrac{z^2}{r^2} = 1$, so $\dfrac{z}{r} = 1$. But $\dfrac{z}{r} = \tan\phi$, so $\phi = \arctan 1 = \dfrac{\pi}{4}$, and we find $\phi \in [0, \pi/4]$. Similarly, $\dfrac{y}{x} = 1 = \arctan\theta$ on the boundary $y = x$, so $\theta \in [0, \pi/4]$. We get

$$\int_0^{pi/4} \int_0^{\pi/4} \int_0^{10} \frac{1}{\rho^2} \rho^2 \sin(\phi) d\rho d\phi d\theta.$$

2. (20 points) Consider the vector field $\vec{F} =< xz, y, z >$ .

   (a) (10 points) Compute $\int\limits_C \vec{F} \cdot d\vec{r}$, where $C$ is the line segment joining $(0,0,0)$ to $(1,2,3)$. **Integrate!** (You may use Maple to compute the integral.)
   **Solution:** Parametrize the line segment as $\vec{r}(t) =< 1,2,3 > t$ on $[0,1]$. We get $\vec{F}(x,y,z) =< xz, y, z >=< 3t^2, 2t, 3t >$ and $\vec{r}'(t) =< 1,2,3 >$ . Thus

$$\begin{aligned} \int\limits_C \vec{F} \cdot d\vec{r} &= \int_0^1 < 3t^2, 2t, 3t > \cdot < 1,2,3 > dt \\ &= \int_0^1 (3t^2 + 4t + 9t) dt \\ &= \frac{15}{2} \text{ (Maple).} \end{aligned}$$

   (b) (5 points) Compute $\int\limits_C \vec{F} \cdot d\vec{r}$, where $C$ is given by $\vec{r}(t) =< t^2, 2t^3, 3t >$ on $[0,1]$. **Integrate!** (You may use Maple to compute the integral.)
   **Solution:** $\vec{F} =< 3t^3, 2t^3, 3t >$ and $\vec{r}'(t) =< 2t, 6t^2, 3 >$, so we get

$$\begin{aligned} \int\limits_C \vec{F} \cdot d\vec{r} &= \int_0^1 < 3t^3, 2t^3, 3t > \cdot < 2t, 6t^2, 3 > \\ &= \int_0^1 (6t^4 + 12t^5 + 9t) dt \\ &= \frac{77}{10} \text{ (Maple).} \end{aligned}$$

   (c) (5 points) Is $\vec{F}$ conservative? Why or why not?
   **Solution:** No; $\vec{F}$ is not conservative since the integrals above are not equal – that means that $\int\limits_C \vec{F} \cdot d\vec{r}$ is not independent of path.

3. (20 points) Let $\vec{F}(x,y) =< 2x + y^2, 2xy >$ .

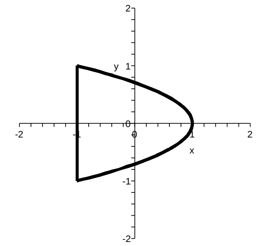
   (a) (10 points) Convince me that $\vec{F}$ is conservative and find a potential function for $\vec{F}$.
   **Solution:** Note that both components are polynomials, so partial derivatives of all orders are continuous everywhere. With $P = 2x + y^2$ and $Q = 2xy$, we see that $Q_x = 2y = P_y$, so $\vec{F}$ is conservative. Call its potential function $f$. Since $f_x = P = 2x+y^2$, we have $f(x,y) = x^2+xy^2+g(y)$ for some function $g$. Since $f_y = Q = 2xy$, we also have $f(x,y) = xy^2 + h(x)$. Choosing $h(x) = x^2$ and $g(y) = 0$, we find the potential function $f(x,y) = x^2 + xy^2$.

   (b) (5 points) Compute $\int\limits_C \vec{F} \cdot d\vec{r}$, where $C$ is the curve parameterized by

   $r(t) =< t^3 \sin(\pi t/2), 1 - (t-1)\sin(t)\cos(t) >$ on $[0,1]$.
   **Solution:** What a horrible curve! Fortunately, $\vec{F}$ is conservative, so the integral is independent of path. We have $\vec{r}(1) =< 1,1 >$ and $\vec{r}(0) =< 0,1 >$ . Therefore, $\int\limits_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,1) = 2$.

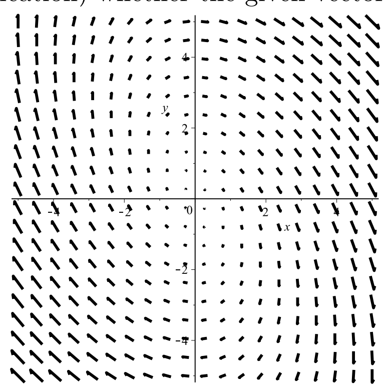
(c) (5 points) Compute $\int_C \vec{F} \cdot d\vec{r}$, where $C$ is the curve shown.

**Solution:** Again, $\vec{F}$ is conservative, so $\int_C \vec{F} \cdot d\vec{r} = 0$ (since $C$ is a closed curve).



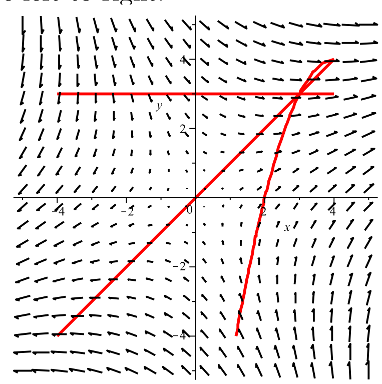

For 4(c)

4. (5 points) Determine (with justification) whether the given vector field is conservative.



**Solution:** Since a swimmer traversing a counterclockwise circle would be pushed along all the way around, such a path integral (around a closed path) would come out positive. Therefore, this vector field cannot be conservative.

5. (5 points) For each of the three given curves in the plane, determine whether $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero. All curves move left to right.



**Solution:** The horizontal line starts out perpendicular to the vector field (so the field contributes nothing) but ends up moving with it, so the net result would be positive. The $y = x$ path has matching interactions with the field on opposite sides of the origin except that the field is oriented in opposite directions in those two quadrants, so the result here would be zero. The rising parabolic path is generally helped by the field, so this one is positive, too.

6. (10 points) **BONUS!!!** Describe a method for determining the surface area of a surface $S$ given by $z = f(x, y)$ over a rectangle $R$. I am looking for an analysis that leads to a double integral (in the manner we have done seven times so far this semester!). You might not find an exact formula, but do your best to describe the process.