Math 249 Final Exam A

December 2013

Name: ________________

Remember to **show all work**. Unsupported solutions will receive **no credit**.
Work on the paper provided.

1 Definitions (15 points)

1. (5 points) State Green's Theorem.

2. (5 points) State Stokes' Theorem.

3. (5 points) State the Divergence Theorem.

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NOTE: You may ask Maple to do any computations you like, but say so.

Captain Willamette and the Bearcat Kid were relaxing by the fireside, having only recently foiled a plot of the evil Dr. Linfield.

"BK, we're going to need to be ready for Dr. Linfield's next move. Let's just have a bit of mental exercise."

1. (10 points) Compute the curl and divergence of $\vec{F} =< z^2 + y, x + z^2, x^2 + y >$ **by hand**. Determine whether $\vec{F}$ is conservative.

2. (10 points) Find the direction of the line of intersection of $4x - 2y + z = 1$ and $3x + 2y - 2z = 4$.

3. (15 points) Fill in each blank with the kind of integral appropriate to it.

single integral C	line integral L	double integral D	surface integral S	triple integral T

(a) ______________ The mass of a hollow sphere of given density.

(b) ______________ The mass of a solid sphere of given density.

(c) ______________ The work done by a given force $\vec{F}$ over a given path C.

(d) ______________ The area under the graph of a function $f(x)$.

(e) ______________ The volume under the graph of a function $f(x, y)$.

Suddenly, the letters WU appeared in the sky as a message came across their WU Monitor. Captain Willamette leapt to his feet, steely muscles tensed for action. The Bearcat Kid was only a shade behind him. "Great leaping bananas, BK! We only have minutes before the evil Dr. Linfield blows this whole mountain to smithereens!" BK paled next to him. "What can we do?"

Captain WU was grim. "We have to defuse the bomb at the top of the mountain."

"But how will we get there in time? It's pitch black outside."

"I happen to know that the mountain is described by $f(x, y) = 1 + x^3 - x^2 - 2y^2$ over the disk $D : x^2 + y^2 \leq 1$. All we have to do is find the global maximum, and we can program the coordinates into the WU Copter."

4. (15 points) Find the top of Mount Calculi. That is, find the global maximum of f over D.

Just as the Dyspeptic Duo programmed the coordinates, the bomb exploded with a roar, tearing the top of the mountain open. "Thundering...err...thunder, Captain WU! He's created a lava lake that will overflow and soon submerge Stokes City under molten metal! What can we do?"

"We just need to know the area of the lava lake so we can bring the right amount of Lava Freeze TM. Too little, and the town is demolished; too much, and we freeze the Earth right down to the core!" The Puny Pair flew the WU Copter around the perimeter of the molten lake and determined that the shore was described by $r(t) =< 2\cos(t) - \cos(2t), 2\sin(t) - \sin(2t) >$.

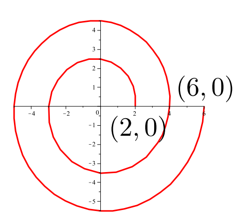
5. (10 points) Find the area of the region bounded by $r(t) =< 2\cos(t) - \cos(2t), 2\sin(t) - \sin(2t) >$ on $[0, 2\pi]$. **Integrate.**

"I just **love** math!"

After cooling the lava lake, our "heroes" overhear Dr. Linfield's newest plot. "Fluctuating functions, Captain WU! He's planning to bombard Salem with nerdium radiation!"

"We'll need to determine the flux so we know how much damage he might do. I have in mind a model for the nerdium field."

6. (10 points) Let the nerdium field be given by $\vec{F} =< 6x, 4y, 3z >$, and let S be the sphere of radius 2 centered at the origin and oriented outward. Compute $\iint_S \vec{F} \cdot d\vec{S}$. **Integrate.** (Note: the path to the right is for Number 7.)



"Snorkeling surfaces! I've never been so happy to get the flux!" "BK, it's different at the south end of town. There's a walking path there I can't parametrize, but we need to calculate the work."

7. (10 points) For the path shown by Number 6, compute $\int_C < 4x + 4y, 4x + 2y > \cdot d\vec{r}$. Start at $(6, 0)$.

"Now let's check on some of the buildings around town. I have some descriptions of them, and we need to find the flux of the nerdium field through them. I've found an even better model for what the nerdium field is.

8. (30 points) Let $\vec{F}(x,y,z) = < e^x, y^2+z^2, 2x+yz >$. **DO NOT INTEGRATE.** Stop once the integral is set up in an appropriate coordinate system. You do not need to simplify integrands. Use any appropriate theorems to set the integral up as you would if you were going to integrate.

   (a) Set up $\iint_S \vec{F}\cdot d\vec{S}$, where S is the closed surface made up of the half-cylinder $x^2+y^2=9$ with $x \ge 0$ and its two bases at $z=0$ and $z=5$. (A half-cylinder with its top, bottom, and flat side.)

   (b) Set up $\iint_S \vec{F}\cdot d\vec{S}$, where S is the part of the surface $z = 4-x^2-y^3$ above $[0,2]\times[3,4]$.

   (c) Set up $\iint_S \text{curl}\vec{F}\cdot d\vec{S}$, where S is the surface made up of the cylinder (with a bottom) surmounted by the frustum of a cone (a cone with the point cut off). The cylinder is given by $x^2+y^2=4$ for $z\in[0,4]$, and the cone is given by $z = 6-\sqrt{x^2+y^2}$ for $z\in[4,5]$. The surface includes the bottom base of the cylinder, so S has three parts.

"It looks to me like the nerdium field is completely harmless. I think Dr. Linfield's evil plan is just to annoy us."

"Fiendishly clever, Captain WU! His plan worked perfectly."

"We must always be alert, for evil is always lurking. Let's wrap up the day with a little more mental exercise."

"Yes, sir, Captain WU!"

"And always remember: NNSNS!*"

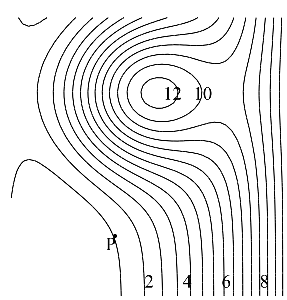
9. (20 points) Let $f(x,y) = 2x^2+3y^2-4xy$.

   (a) Compute the directional derivative of f at $(1,2,6)$ in the direction of $< 3,-4 >$.

   (b) Determine the direction of greatest increase of f at $(1,2,6)$ and the rate of increase in that direction.

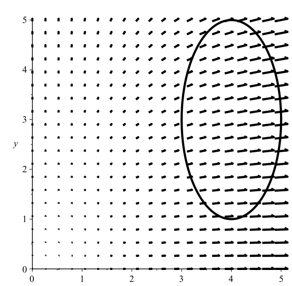
10. (10 points) Let C be the space curve given by $r(t) = < 3t, t^2, 2t^3 >$. Find parametric equations of the tangent line to C at the point $(3,1,2)$.

11. (10 points) Consider the contour map shown to the right. Starting at P, sketch the path that climbs the fastest and indicate where the "top" is.



12. (15 points) Verify Green's Theorem by computing $\vec{F}\cdot d\vec{r}$ in two ways, where C is the unit circle and $\vec{F} = < y^2, xy^2 >$.

13. (20 points) Consider the vector field $\vec{F}$ and closed curve C shown to the right.



   (a) Determine, with justification, whether $\oint_C \vec{F}\cdot d\vec{r}$ is positive, negative, or zero.

   (b) Determine, with justification, whether the flux of $\vec{F}$ through C is positive, negative, or zero.

* *Non nobis solum nati sumus.*