Math 249 Final Exam A Solutions

December 2013

Name: ________________

Remember to **show all work**. Unsupported solutions will receive **no credit**.
Work on the paper provided.

1 Definitions (15 points)

1. (5 points) State Green's Theorem.

   Solution: See your text or notes.

2. (5 points) State Stokes' Theorem.

   Solution: See your text or notes.

3. (5 points) State the Divergence Theorem.

   Solution: See your text or notes.

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NOTE: You may ask Maple to do any computations you like, but say so.

1. (10 points) Compute the curl and divergence of $\vec{F} =< z^2 + y, x + z^2, x^2 + y >$ **by hand**. Determine whether $\vec{F}$ is conservative.

   Solution: curl$\vec{F} = \nabla \times \vec{F} =< 1-2z, -(2x-2z), 1-1 >=< 1-2z, 2z-2x, 0 >$. Div$\vec{F} = 0+0+0 = 0$. Since the curl of $\vec{F}$ is not $\vec{0}$, $\vec{F}$ is not conservative.

2. (10 points) Find the direction of the line of intersection of $4x - 2y + z = 1$ and $3x + 2y - 2z = 4$.

   Solution: The line of intersection lies in both planes and is therefore perpendicular to both normals, which are $< 4, -2, 1 >$ and $< 3, 2, -2 >$. Thus, $\mathbf{v} =< 4, -2, 1 > \times < 3, 2, -2 >=< 2, 11, 14 >$ is the direction vector.

3. (15 points) Fill in each blank with the kind of integral appropriate to it.

single integral	line integral	double integral	surface integral	triple integral
C	L	D	S	T

   (a) S The mass of a hollow sphere of given density.

   (b) T The mass of a solid sphere of given density.

   (c) L The work done by a given force $\vec{F}$ over a given path C.

   (d) C The area under the graph of a function $f(x)$.

   (e) D The volume under the graph of a function $f(x, y)$.

   "I happen to know that the mountain is described by $f(x, y) = 1 + x^3 - x^2 - 2y^2$ over the disk $D : x^2 + y^2 \leq 1$. All we have to do is find the global maximum, and we can program the coordinates into the WU Copter."

4. (15 points) Find the top of Mount Calculi. That is, find the global maximum of f over D.

   Solution: We need the critical points and the boundary behavior. We have $f_x = 3x^2 - 2x$ and $f_y = -4y$, so $y = 0$ and $x = 0$ or $2/3$. $\boxed{f(0,0) = 1 \text{ and } f(2/3, 0) = 23/27.}$

   On the boundary, we have $x^2 + y^2 = 1$. We could use $y^2 = 1 - x^2$ or try Langrange multipliers. I will do both so you have both examples.

   (1) With $y^2 = 1-x^2$, we have $g(x) = f(x, \pm\sqrt{1-x^2}) = 1+x^3-x^2-2(1-x^2) = x^3+x^2-1$ for $x \in [-1, 1]$. We get $g'(x) = 3x^2+2x = 0$ for $x = 0$ or $x = -2/3$. $\boxed{g(-1) = -1, g(-2/3) = -23/27, g(0) = -1, \text{ and } g(1) = 1.}$

   Out of all of the boxed values, the largest is 1 (occurring at (1,0) and (0,0)) and the smallest is -1 (occurring at $(-1, 0)$ and $(0, \pm 1)$).

   (2) Let $g(x, y) = x^2 + y^2$. Then

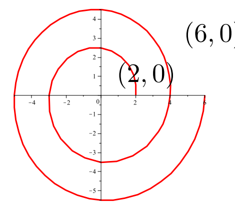
$$\begin{aligned}\nabla f = \lambda \nabla g &\implies < 3x^2 - 2x, -4y >= \lambda < 2x, 2y > \\ &\implies 3x^2 - 2x = 2\lambda x \text{ and } -4y = 2\lambda y.\end{aligned}$$

Thus $y = 0$ or $\lambda = -2$ and $x = 0$ or $x = \dfrac{2}{3}(\lambda + 1)$. From $x^2 + y^2 = 1$, we get $y = 0 \implies x = \pm 1$, giving $\boxed{f(1,0) = 1, f(-1,0) = -1.}$ If $\lambda = -2$, we get $x = 0$ and $y = \pm 1$ (giving $\boxed{f(0, \pm 1) = -1}$) or $x = -2/3$ and $y = \pm\sqrt{5}/3$ (giving $\boxed{f(-2/3, \pm\sqrt{5}/3) = -23/27}$ (see above)). Thus, comparing our boxed numbers, we get the same result as previously.

5. (10 points) Find the area of the region bounded by $r(t) =< 2\cos(t) - \cos(2t), 2\sin(t) - \sin(2t) >$ on $[0, 2\pi]$. **Integrate.**

   Solution: $A = \int_C x dy = \displaystyle\int_0^{2\pi} (2\cos(t) - \cos(2t))(2\cos(t) - 2\cos(2t))dt = 5\pi$ (via Maple). This uses the same idea you used on the project.

6. (10 points) Let the nerdium field be given by $\vec{F} =< 6x, 4y, 3z >$, and let S be the sphere of radius 2 centered at the origin and oriented outward. Compute $\iint_S \vec{F} \cdot d\vec{S}$. **Integrate.** (Note: the path to the right is for Number 7.)



   Solution: By the Divergence Theorem, we have $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F} dV$ $= \iiint_E 13 dV = 13V(E) = 13 \cdot \dfrac{4}{3}\pi \cdot 2^3$.

7. (10 points) For the path shown by Number 6, compute $\int_C < 4x + 4y, 4x + 2y > \cdot d\vec{r}$. Start at $(6, 0)$.

   Solution: Notice that $P = 4x + 4y$ and $Q = 4x + 2y$, so $Q_x = 4 = P_y$. Since these partials are continuous everywhere in $\mathbb{R}^2$, $\vec{F}$ is conservative, so we just need a potential function and then we can use the FTCLI. With $f_x = 4x + 4y$, we find $f = 2x^2 + 4xy + g(y)$, and then $f_y = 4x + g'(y)$. Since we already know that $f_y = Q = 4x + 2y$, we find $g'(y) = 2y$, so $g(y) = y^2 + C$. Take $f(x, y) = 2x^2 + 4xy + y^2$. Then $\int_C < 4x + 4y, 4x + 2y > \cdot d\vec{r} = f(2, 0) - f(6, 0) = 8 - 72 = -64$.

8. (30 points) Let $\vec{F}(x, y, z) =< e^x, y^2 + z^2, 2x + yz >$. **DO NOT INTEGRATE.** Stop once the integral is set up in an appropriate coordinate system. You do not need to simplify integrands. Use any appropriate theorems to set the integral up as you would if you were going to integrate.

   (a) Set up $\iint_S \vec{F} \cdot d\vec{S}$, where S is the closed surface made up of the half-cylinder $x^2 + y^2 = 9$ with $x \geq 0$ and its two bases at $z = 0$ and $z = 5$. (A half-cylinder with its top, bottom, and flat side.)
   Solution: Using the Divergence Theorem, we have

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iiint_E \text{div}\vec{F} dV \\ &= \iiint_E (e^x + 3y) dV \\ &= \int_{-\pi/2}^{\pi/2} \int_0^9 \int_0^5 (e^{r\cos\theta} + 3r\sin\theta) r dz dr d\theta.\end{aligned}$$

   (b) Set up $\iint_S \vec{F} \cdot d\vec{S}$, where S is the part of the surface $z = 4 - x^2 - y^3$ above $[0, 2] \times [3, 4]$.

   Solution: We have $\vec{F} =< P, Q, R >$ and $z = g(x, y)$, so $iint_S \vec{F} \cdot d\vec{S} = \displaystyle\int_0^2 \int_3^4 (-e^x(-2x) - (y^2 + (4 - x^2 - y^3)^2)(-3y^2) + 2x + y(4 - x^2 - y^3))dydx$.

   (c) Set up $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$, where S is the surface made up of the cylinder (with a bottom) surmounted by the frustum of a cone (a cone with the point cut off). The cylinder is given by $x^2 + y^2 = 4$ for $z \in [0, 4]$, and the cone is given by $z = 6 - \sqrt{x^2 + y^2}$ for $z \in [4, 5]$. The surface includes the bottom base of the cylinder, so S has three parts.

   Solution: This surface is complicated, but closed. We can apply the Divergence Theorem here, but since divcurl$\vec{F} = 0$ (for any vector field, not just this one), the integral is 0.

9. (20 points) Let $f(x, y) = 2x^2 + 3y^2 - 4xy$.

   (a) Compute the directional derivative of f at $(1, 2, 6)$ in the direction of $< 3, -4 >$.

   Solution: $\nabla f =< 4x-4y, 6y-4x >$; at $(1, 2)$, this is $\nabla f(1, 2) =< -4, 8 >$. Our $u = \frac{1}{5} < 3, -4 >$, so $D_u f(1, 2) =< -4, 8 > \cdot \frac{1}{5} < 3, -4 >= -\frac{44}{5}$.

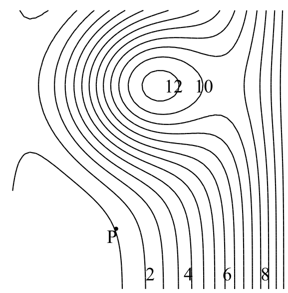
   (b) Determine the direction of greatest increase of f at $(1, 2, 6)$ and the rate of increase in that direction.

   Solution: That direction is the direction of the gradient: $< -4, 8 >$. The maximum rate of increase is the magnitude of this: $4\sqrt{5}$.

10. (10 points) Let C be the space curve given by $r(t) =< 3t, t^2, 2t^3 >$. Find parametric equations of the tangent line to C at the point $(3, 1, 2)$.

    Solution: With $(3, 1, 2) = (3t, t^2, 2t^3)$, we have $t = 1$. Now $r'(t) =< 3, 2t, 6t^2 >$ and $r'(1) =< 3, 2, 6 >$. Thus our tangent line is given by $x = 3 + 3t, y = 1 + 2t, z = 2 + 6t$.

11. (10 points) Consider the contour map shown to the right. Starting at P, sketch the path that climbs the fastest and indicate where the "top" is.



    Tricky for me to actually draw, but you want a path that goes from P to inside the 12 contour line in the middle. It should hit each curve along the way at right angles to that curve's tangent line.

12. (15 points) Verify Green's Theorem by computing $\oint_C \vec{F} \cdot d\vec{r}$ in two ways, where C is the unit circle and $\vec{F} =< y^2, xy^2 >$.

    Solution: Green's Theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (y^2 - 2y)dA = \int_0^{2\pi} \int_0^1 (r^2 \sin^2\theta - 2r\sin\theta) r dr d\theta \overset{M}{=} \frac{\pi}{4}$.

    Parametrization: $\vec{r} =< \cos(\theta), \sin(\theta) >$ and $\vec{r}' =< -\sin(\theta), \cos(\theta) >$. Thus $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} < \sin^2\theta, \cos(\theta)\sin^2\theta > \cdot < -\sin(\theta), \cos(\theta) >= -\sin^3(\theta) + \cos^2(\theta)\sin^2(\theta) \overset{M}{=} \frac{\pi}{4}$, as we hoped!

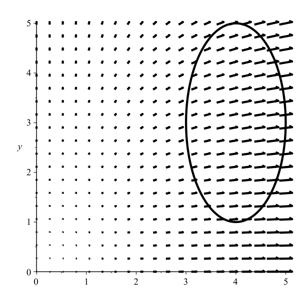
13. (20 points) Consider the vector field $\vec{F}$ and closed curve C shown to the right.

    (a) Determine, with justification, whether $\oint_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero.

    Solution: It looks like I neglected to include a direction, so let's say we're going counterclockwise. It appears to me that we have no net contribution from the horizontal components since each little bit of horizontal help is counteracted by a corresponding but of horizontal hindrance. Likewise, the vertical components all seem to match up, so I think the integral should be 0.

    (b) Determine, with justification, whether the flux of $\vec{F}$ through C is positive, negative, or zero.

    Solution: The flux here should be positive since larger arrows are leaving from C than are entering into C.



* *Non nobis solum nati sumus.*