Solutions to Homework Assignment 10

1. $r'(t) = < 2\cos t, 5, -2\sin t >$, so $L = \int_{-10}^{10} \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2}dt = \int_{-10}^{10} \sqrt{29}\ dt = \sqrt{29}t\Big|_{-10}^{10} = \sqrt{29}(10-(-10)) = 20\sqrt{29}$.

3. $r'(t) = < \sqrt{2}, e^t, -e^{-t} >$, so

$$\begin{aligned} L &= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}dt \\ &= \int_0^1 \sqrt{(e^t + e^{-t})^2}dt \\ &= \int_0^1 (e^t + e^{-t})dt \\ &= e^t - e^{-t}\big|_0^1 \\ &= e - e^{-1}. \end{aligned}$$

13. $r'(t) = < 2, -3, 4 >$, so we have $s(t) = \int_0^t \sqrt{2^2 + (-3)^2 + 4^2}\ du = \sqrt{29}u\Big|_0^t = \sqrt{29}t$. Therefore, $t = \frac{1}{\sqrt{29}}s$. We get $r(t(s)) = \left\langle \frac{2s}{\sqrt{29}}, 1 - \frac{3s}{\sqrt{29}}, 5 + \frac{4s}{\sqrt{29}} \right\rangle$. Notice that with this, if $s = 1$, the point on the curve (a line) is $\left\langle \frac{2}{\sqrt{29}}, 1 - \frac{3}{\sqrt{29}}, 5 + \frac{4}{\sqrt{29}} \right\rangle$. The distance of this along the curve from the point $(0, 1, 5)$ (where we started) is $\sqrt{4/29 + 9/29 + 16/29} = 1$ since the curve is just a line, and this is the value of s. This is the point of the reparametrization: the arc length is exactly the value of the parameter. (Try this for $s = 2$ and $s = 7$.)

15. Since $|r'(t)| = |< 3\cos t, 4, -3\sin t >| = 5$, $s(t) = \int_0^t 5\ du = 5u\Big|_0^t = 5t$. Thus $t = \frac{s}{5}$. Reparametrization gives us $r(t(s)) = < 3\sin(s/5), 4s/5, 3\cos(s/5) >$.

17. (a) $r'(t) = < 2\cos t, 5, -2\sin t >$ and $|r'(t)| = \sqrt{29}$, so $T(t) = \frac{1}{\sqrt{29}} < 2\cos t, 5, -2\sin t >$. $T'(t) = \frac{1}{\sqrt{29}} < -2\sin t, 0, -2\cos t >$ and $|T'(t)| = \frac{2}{\sqrt{29}}$. Thus $N(t) = < -\sin t, 0, -\cos t >$.

    (b) We have $\kappa = \frac{2}{29}$.

19. (a) $r'(t) = < \sqrt{2}, e^t, -e^{-t} >$ and $|r'(t)| = e^t + e^{-t}$ (from Number 3). Thus

$$\begin{aligned} T(t) &= \frac{1}{e^t + e^{-t}} < \sqrt{2}, e^t, -e^{-t} > \\ &= \frac{1}{e^{2t} + 1} < \sqrt{2}e^t, e^{2t}, -1 >. \end{aligned}$$

This gives

$$T'(t) = \left\langle \frac{\sqrt{2}[e^t(e^{2t}+1) - 2e^{2t}e^t)]}{(e^{2t}+1)^2}, \frac{2e^{2t}(e^{2t}+1) - 2e^{2t}e^{2t}}{(e^{2t}+1)^2}, \frac{2e^{2t}}{(e^{2t}+1)^2} \right\rangle = \frac{\sqrt{2}e^t}{(e^{2t}+1)^2} < 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t >.$$

Now

$$\begin{aligned} |T'(t)| &= \frac{\sqrt{e^t}}{(e^{2t}+1)^2} \sqrt{(1-e^{2t})^2 + 2e^{2t} + 2e^{2t}} \\ &= \frac{\sqrt{2}e^t}{(1+e^{2t})^2}(1 + e^{2t}), \end{aligned}$$

so $N(t) = \frac{1}{1+e^{2t}} < 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t >$.

(b) $\kappa(t) = \dfrac{\sqrt{2}e^t}{1+e^{2t}}\dfrac{e^t}{e^{2t}+1} = \dfrac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2}$.

37. Notice that the blue curve has two points on its left half where the curvature would be the same. In fact, at the inflection point the curvature would be 0 (see Box 11). Since the red curve does not have zeroes at the blue curve's inflection points, the red cannot be the blue's curvature. Therefore, the blue is the red's curvature.

43. $r'(t) =< 2t, 2t^2, 1 >$, so $T(t) = \dfrac{< 2t, 2t^2, 1 >}{\sqrt{4t^2+4t^4+1}} = \dfrac{< 2t, 2t^2, 1 >}{2t^2+1}$. At $r(t) =< 1, 2/3, 1 >$, we have $t = 1$, so $T(1) =< 2/3, 2/3, 1/3 >$. $T'(t) = \dfrac{1}{(2t^2+1)^2} < 2(2t^2+1) - 4t(2t), 4t(2t^2+1) - 4t(2t^2), -4t >$. $T'(1) = \dfrac{1}{9} < -2, 4, -4 >$, so $N(1) = \dfrac{1}{3} < -1, 2, -2 >$. $B(1) = T(1) \times N(1) = \dfrac{1}{9} < -6, 3, 6 >= \dfrac{1}{3} < -2, 1, 2 >$.

45. At $(0, \pi, -2)$, $t = \pi$. We need T, N, and B. $r'(t) =< 6\cos 3t, 1, -6\sin 3t >$, so $|r'(t)| = \sqrt{37}$. Thus $T(t) = \dfrac{1}{\sqrt{37}} < 6\cos 3t, 1, -6\sin 3t >$. Now $T'(t) = \dfrac{1}{\sqrt{37}} < -18\sin 3t, 0, -18\cos 3t >$, so $N(t) =< -\sin 3t, 0, -\cos 3t >$. At $t = \pi$, we get $T(\pi) = \dfrac{1}{\sqrt{37}} < -6, 1, 0 >$ and $N(\pi) =< 0, 0, 1 >$. Thus $B(\pi) = \dfrac{1}{\sqrt{37}} < -6, 1, 0 > \times < 0, 0, 1 >= \dfrac{1}{\sqrt{37}} < 1, 6, 0 >$.
For equations of planes, we only care about the direction of the normal, so I am going to use $< -6, 1, 0 >$ as the normal to the normal plane: $-6x + y = \pi$.
For the osculating plane, I will use $< 1, 6, 0 >$: $x + 6y = 6\pi$.