Solutions to Homework Assignment 13

MATH 249-01 and -02

Section 14.2, Page 908

1, 5, 6, 7, 9, 11, 12, 14, 15, 16, 17, 18, 20, 27, 29, 30, 31, 34

1. We can't say anything unless we know f is continuous. If so, then $f(3,1) = 6$.
5. $\lim_{(x,y)\to(5,-2)} (x^5 + 4x^3y - 5xy^2) = 5^5 + 4(5)^3(-2) - 5(5)(-2)^2 = 2025$.
7. If we approach $(0,0)$ along $y = x$, we get $\frac{1}{2}$. If we approach along $y = 2x$, we get $\frac{1}{5}$. Therefore, the limit does not exist.
9. If we approach along $y = x$, we get $\frac{x^2 \cos x}{4x^2} \to \frac{1}{4}$. If we approach along $y = -x$, we get $\frac{-x^2 \cos x}{4x^2} \to -\frac{1}{4}$. Therefore, the limit does not exist.
11. This has the look to me of, roughly, x^2/x, which I would expect to approach 0. I will guess 0 as my limit and try to prove that I am right.

    Let $\epsilon > 0$ be given. I need $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, then $\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \epsilon$. Notice that $(x-y)^2 \geq 0$, so $x^2 - 2xy + y^2 \geq 0$. Therefore, $xy \leq \frac{1}{2}(x^2+y^2)$. We get

    $$\begin{aligned}\left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leq \frac{1}{2}\frac{x^2+y^2}{\sqrt{x^2+y^2}} \\ &= \frac{1}{2}\sqrt{x^2+y^2}.\end{aligned}$$

    Since I want the first expression less than ϵ, I will choose $\delta = 2\epsilon$. Now if $0 < \sqrt{x^2+y^2} < \delta$, then $\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{1}{2}\sqrt{x^2+y^2} < \frac{1}{2}\delta = \frac{1}{2}(2\epsilon) = \epsilon$, as desired. Therefore, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

    We could also have used the squeeze theorem: since $|x| \leq \sqrt{x^2+y^2}$, $\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq |y|$, which approaches 0 as $(x,y) \to (0,0)$.
15. I will rationalize the denominator:

    $$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\,\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \\ &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2} \\ &= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1) \\ &= 2.\end{aligned}$$
16. If we approach along $x = y^4$, we get $\frac{y^8}{2y^8} = \frac{1}{2}$. If we approach along $y = x$, we get $\frac{y^5}{y^2+y^8} = \frac{y^3}{1+y^6} \to 0$. Therefore, this limit does not exist.
17. Since the function is continuous at $(3,0,1)$, we may simply substitute to get $e^0 \sin(\pi/2) = 1$.
18. If we approach along the x-axis, we get $\frac{x^2}{x^2} = 1$. If we approach along the y-axis, we get $\frac{2y^2}{y^2} = 2$. Therefore, this limit does not exist.
27. F is continuous provided $x \neq \ln y^2$.
29. $\arctan t$ is continuous everywhere, so this only depends on the inside function. $x + \sqrt{y}$ is continuous on its domain, which is the upper half-plane $\{(x,y)|y \geq 0\}$.
31. $\ln t$ is continuous on $(0,\infty)$, so we need $x^2 + y^2 - 4 > 0$. This is the exterior of the circle $x^2 + y^2 = 4$ (not including the circle itself).

34. We need $x + y + z \geq 0$. The plane $x + y + z = 0$ passes through the origin and has normal $< 1, 1, 1 >$. To be on the correct side of that plane, we need to be on the same side as this normal; that is, the domain is the half-space containing the first octant plus this plane.