Solutions to Homework Assignment 18

MATH 249

Section 14.7, Page 930 Stewart 6e

1, 3, 4, 5, 8, 11, 15, 29, 33, 35, 40, 43, 47, 49

1. (a) Since $f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = 7 > 0$ and $f_{xx}(1,1) > 0$, $f(1,1)$ is a local minimum of f by the second derivatives test.

   (b) This time, $f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = -1 < 0$, so f has a saddle point at $(1,1)$.

3. Since the level curves at the origin are tangent to the axes, heading in the direction of either keeps you on the level curve. Thus $f_x(0,0) = f_y(0,0) = 0$, so $(0,0)$ is a critical point. However, since moving along the line $y = x$ causes a decrease in f and moving along $y = -x$ causes an increase, $(0,0)$ is a saddle point.
   The loop closing in around (1,1) also indicates a critical point: from (1,1), every direction leads up! Thus $f(1,1)$ is a local minimum.
   $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$. For these to equal zero, we need $y = x^2$ and $y^2 - x = x^4 - x = 0$. This gives $x(x-1)(x^2+x+1) = 0$, so $x = 0$ or $x = 1$. If $x = 0$, then $y = 0$; if $x = 1$, then $y = 1$. Thus, we have the two critical points $(0,0)$ and $(1,1)$. (So far, so good!) $f_{xx} = 6x, f_{xy} = -3$, and $f_{yy} = 6y$, so $D = 36xy - 9$. $D(0,0) = -9 < 0$, so f has a saddle point at $(0,0)$, as we found. $D(1,1) = 27 > 0$, so f has a local minimum at $(1,1)$, as we found. Yay us!

4. Based on our experience in 3, I will go out on a limb and guess that f has saddle points at $(-1,0), (1,1)$, and $(1,-1)$, local minima at $(-1,1)$ and $(-1,-1)$, and a local maximum at $(1,0)$. Let's see!
   $f_x = 3-3x^2$ and $f_y = -4y+4y^3$. For these to be zero we need $x = \pm 1$ and $y = 0, \pm 1$. The combinations are $(1,0), (1,1), (1,-1), (-1,0), (-1,1)$, and $(-1,-1)$. Note that these are the points we found above. $f_{xx} = -6x, f_{yy} = -4+12y^2$, and $f_{xy} = 0$. Thus $D(x,y) = 24x - 72xy^2 = 24x(1-3y^2)$. This is positive for $(1,0), (-1,1)$, and $(-1,-1)$. The other three points are saddle points. $f_{xx} > 0$ for $(-1,1)$ and $(-1,-1)$, so these give local minima. $f_{xx}(1,0) > 0$, so $f(1,0)$ is a local maximum.

5. $f_x = -2 - 2x$ and $f_y = 4 - 8y$. The only critical point is thus $(-1, 1/2)$. $f_{xx} = -2, f_{yy} = -8$, and $f_{xy} = 0$, so $D = 16$. Since $D(-1,1/2) > 0$ and $f_{xx}(-1,1/2) < 0$, $f(-1,1/2)$ is a local maximum.

15. $f_x = 2xe^{y^2-x^2} - 2x(x^2+y^2)e^{y^2-x^2} = 2x(1-x^2-y^2)e^{y^2-x^2}$ and $f_y = 2ye^{y^2-x^2} + 2y(x^2+y^2)e^{y^2-x^2} = 2y(1+x^2+y^2)e^{y^2-x^2}$. f_x is zero for $x = 0$ or $x^2+y^2 = 1$ and f_y is zero only for $y = 0$. Thus $(0,0), (1,0)$, and $(-1,0)$ are the critical points.
    $f_{xx} = (2-6x^2-2y^2)e^{y^2-x^2} - 2xe^{y^2-x^2}2x(1-x^2-y^2), f_{yy} = (2+2x^2+6y^2)e^{y^2-x^2} + 2ye^{y^2-x^2}2y(1+x^2+y^2)$, and $f_{xy} = -4xye^{y^2-x^2} + 4xy(1-x^2-y^2)e^{y^2-x^2}$. Mercy!
    Thus $D(0,0) = (2)(2) - 0 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $f(0,0) = 0$ is a local minimum.
    $D(1,0) = (-4e^{-1})(4e^{-1}) - 0 = -16e^{-2} < 0$, so f has a saddle point at $(1,0)$. $D(-1,0) = D(1,0)$, so f also has a saddle point at $(-1,0)$.

29. $f_x = 4, f_y = -5$, so there are no critical points. The boundary consists of the three line segments $L_1 : y = 0, 0 \le x \le 2, L_2 : x = 0, 0 \le y \le 3$, and $L_3 : 3x + 2y = 6$ for $0 \le x \le 2$. On L_1, $f(x,y) = 1 + 4x$, which is increasing. It attains its maximum value of 9 at $x = 2$ and its minimum value of 1 at $x = 0$.
    On $L_2, f(x,y) = 1 - 5y$, which is decreasing. It attains its maximum value of 1 at $y = 0$ and its minimum value of -14 at $y = 3$.
    On $L_3, y = -\frac{3}{2}x+3$, so $f(x,y) = 1+4x-5\left(-\frac{3}{2}x+3\right) = \frac{23}{2}x - 14$. This is increasing, so its maximum value of 9 is at $x = 2$ and its minimum value of -14 is at $x = 0$. Therefore, the absolute maximum of f on D is 9 and occurs at $(2,0)$, and the absolute minimum is -14 and occurs at $(0,3)$.

33. $f_x = 4x^3 - 4y, f_y = 4y^3 - 4x$. If these are both zero, then $y = x^3$, so $x^9 - x = 0$. This factors as $x(x^8-1) = x(x^4-1)(x^4+1) = x(x^2-1)(x^2+1)(x^4+1) = x(x-1)(x+1)(x^2+1)(x^4+1)$. The zeros are $x = 0, \pm 1$. The critical points are therefore $(0,0), (1,1)$, and $(-1,-1)$ since $y = x^3$.
    We have four boundary curves: $L_1 : x = 0, 0 \le y \le 2, L_2 : y = 2, 0 \le x \le 3, L_3 : x = 3, 0 \le y \le 2$, and $L_4 : y = 0, 0 \le x \le 3$.
    On $L_1, f(x,y) = y^4 + 2$. This is increasing, so the maximum is $f(0,2) = 18$ and the minimum is

$f(0,0) = 2$.
On L_2, let $g(x) = f(x,y) = x^4 - 8x + 18$. $g'(x) = 4x^3 - 8$, which has a zero at $x = \sqrt[3]{2}$. $g(0) = 18, g(\sqrt[3]{2}) = 18 - 6\sqrt[3]{2}$, and $g(3) = 75$, so the maximum is $f(3,2) = 75$ and the minimum is $f(\sqrt[3]{2}, 2) = 18 - 6\sqrt[3]{2}$.
On L_3, let $h(y) = f(3,y) = y^4 - 12y + 83$. $h'(y) = 4y^3 - 12$, which has a zero at $y = \sqrt[3]{3}$. $h(0) = 83, h(\sqrt[3]{3}) = 83 - 9\sqrt[3]{3}$, and $h(2) = 75$, so the maximum is $f(3,0) = 83$ and the minimum is $h(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3}$.
On L_4, $f(x,y) = x^4 + 2$. The maximum is $f(3,0) = 83$ and the minimum is $f(0,0) = 2$.
At the critical points, we have $f(0,0) = 2$ and $f(1,1) = 0$; the critical point $(-1,-1)$ is outside the region.
Comparing all of these numbers, we see that the absolute maximum of f on D is $f(3,0) = 83$ and the absolute minimum is $f(1,1) = 0$.
Whew!

35. $f_x = 6x^2$ and $f_y = 4y^3$, so the only critical point is $(0,0)$. The boundary is $x^2 + y^2 = 1$, so $y^2 = 1 - x^2$. Therefore, on the boundary curve C, we have $g(x) = f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$. $g'(x) = 4x^3 + 6x^2 - 4x = 2x(2x^2 + 3x - 2) = 2x(2x-1)(x+2)$. The critical points on the boundary are at $x = 0, x = 1/2$, and $x = -2$. Since $x = -2$ does not correspond to a point on the circle, we only have the first two. We get the points $(0,-1), (0,1), (1/2, -\sqrt{3}/2), (1/2, \sqrt{3}/2)$. We also need to check the endpoints of the interval $[-1,1]$.
Now $f(0,0) = 0, f(0,-1) = f(0,1) = 1, f(1/2, \pm\sqrt{3}/2) = \dfrac{13}{16}, f(-1,0) = -2$, and $f(1,0) = 3$. The absolute maximum is therefore $f(1,0) = 2$, and the absolute minimum is $f(-1,0) = -2$.

43. Let x, y, and z be the numbers. We have $x + y + z = 100$, so $z = 100 - x - y$. Let $f(x,y) = xy(100 - x - y) = 100xy - x^2y - xy^2$. We are required to have $x, y, z \geq 0$, so our domain has boundaries $x = 0, y = 0$, and $x + y = 100$. Note that $f(x,y) = 0$ on all of the boundaries.
$f_x = 100y - 2xy - y^2 = y(100 - 2x - y)$ and $f_y = 100x - x^2 - 2xy = x(100 - x - 2y)$. We need these simultaneously zero to find our critical points. If $y = 0$, then $f_y = 100x - x^2$, which is zero for $x = 0$ and $x = 100$. We get the critical points $(0,0)$ and $(100,0)$. If $y \neq 0$, then we must have $y = 100 - 2x$ so the second factor of f_x is zero. This gives $f_y = x(100 - x - 2(100 - 2x)) = x(3x - 100)$. This is zero for $x = 0$ or $x = 100/3$. $x = 0$ is on a boundary, so we have already dealt with it. $x = 100/3$ gives $y = 100/3$ and $f(x,y) = 100^3/27$. This is clearly our maximum! The numbers are thus $\dfrac{100}{3}, \dfrac{100}{3}, \dfrac{100}{3}$.

47. The plane $x + 2y + 3z = 6$ tells us how high the box can reach: $z = 2 - \dfrac{1}{3}x - \dfrac{2}{3}y$. The volume is therefore $V(x,y) = xy\left(2 - \dfrac{x}{3} - \dfrac{2y}{3}\right) = 2xy - \dfrac{x^2y}{3} - \dfrac{2xy^2}{3}$. The boundaries are $x = 0, y = 0$, and $2 - x/3 - 2y/3 = 0$. All of these give $V = 0$.
$V_x(x,y) = 2y - \dfrac{2xy}{3} - \dfrac{2y^2}{3} = \dfrac{1}{3}y(6 - 2x - 2y)$ and $V_y(x,y) = 2x - \dfrac{x^2}{3} - \dfrac{4xy}{3} = \dfrac{1}{3}x(6 - x - 4y)$. For V_x to be zero, we must have $y = 0$ or $y = 3 - x$. If $y = 0$, then $V = 0$. If $y = 3 - x$, then $V_y = \dfrac{1}{3}x(6 - x - 4(3 - x)) = \dfrac{1}{3}x(3x - 6)$, so $x = 0$ or $x = 2$. The critical points are $(0,0)$ and $(2,1)$.
$V(0,0) = 0, V(2,1) = \dfrac{4}{3}$, which is our maximum volume.

49. If x, y, and z are the sides, then the volume is xyz and the sum of the side lengths is $4(x+y+z) = 4c$. (I'm going to use 4x to avoid fractions!) Thus $z = c - x - y$ and $V(x,y) = xy(c - x - y) = cxy - x^2y - xy^2$. The boundaries are $x = 0, y = 0$, and $x + y = c$, and V is zero along all of them.
$V_x(x,y) = cy - 2xy - y^2 = y(c - 2x - y)$ and $V_y = cx - x^2 - 2xy = x(c - x - 2y)$. $V_x = 0$ gives $y = 0$ or $y = c - 2x$. As before, $y = 0$ will not get us a maximum for this function. $y = c - 2x$ gives $V_y = x(c - x - 2(c - 2x)) = x(3x - c)$, so $x = 0$ or $x = c/3$. Only the latter is interesting, so the only critical point we need to consider is $(c/3, c/3)$. This gives dimensions $\dfrac{c}{3}$ by $\dfrac{c}{3}$ by $\dfrac{c}{3}$. This is pretty reasonable since there is no clear preference given to length, width, or height. If you used the book's c instead of mine, you should get an edge length of $\dfrac{c}{12}$ since my c is four times the book's.