Solutions to Homework Assignment 29

MATH 249

Section 16.3 Stewart 6e, Page 1053

1-9, 11-23, 29-32

1. Since we are integrating a continuous gradient along a smooth path, the result is independent of path. We get $50 - 10 = 40$.
2. Since $r'(t) =< 2t, 3t^2 + 1 >$, C is smooth. We are told that ∇f is continuous, so the integral is just $f(r(1)) - f(r(0)) = f(2,2) - f(1,0) = 9 - 3 = 6$.
3. Since P and Q are defined on $\mathbb{R}^2$ and $Q_x = -3 = P_y$ (and the other partials are also continuous), F is conservative. Thus, there is a function f such that $\nabla f(x,y) =< 2x - 3y, -3x + 4y - 8 >$. Integrating the first component with respect to x and the second with respect to y gives $f(x,y) = x^2 - 3xy + g(y) = -3xy + 2y^2 - 8y + h(x)$. Thus $h(x) = x^2 + C_1$ and $g(y) = 2y^2 - 8y + C_2$ for some constants C_1 and C_2. We have $f(x,y) = x^2 - 3xy + 2y^2 - 8y + C$ for some constant C.
6. $Q_x = 4y, P_y = -4y$, so F is not conservative.
7. $Q_x = e^x + \cos y = P_y$. The domain of F is $\mathbb{R}^2$ and its first partials are continuous, so F is conservative. $\nabla f(x,y) =< ye^x + \sin y, e^x + x\cos y >$, so $f(x,y) = ye^x + x\sin y + g(y) = ye^x + x\sin y + h(x)$. This gives $f(x,y) = ye^x + x\sin y + C$.
11. The domain of F is $\mathbb{R}^2$, the first partials are continuous, and $Q_x = 2x = P_y$, so F is conservative. Thus, $\int_C F \cdot dr$ is independent of path. $f(x,y) = x^2y$ is a potential function for F, so the integral is $f(3,2) - f(1,2) = 18 - 2 = 16$.
14. $f_x(x,y) = \dfrac{y^2}{1+x^2}$, so $f(x,y) = y^2 \arctan x + g(y)$. With this f, we have $f_y(x,y) = 2y \arctan x + g'(y)$, so $g'(y) = 0$. Thus we let $f(x,y) = y^2 \arctan x$. Now the endpoints of the curve are $(1,2)$ and $(0,0)$, so we get $f(1,2) - f(0,0) = 4(\pi/4) - 0 = \pi$.
15. $\nabla f(x,y,z) =< yz, xz, xy + 2z >$, so $f(x,y,z) = xyz + g(y,z) = xyz + h(x,z) = xyz + z^2 + k(x,y)$. We may take $f(x,y,z) = xyz + z^2$. The integral has the value $f(4,6,3) - f(1,0,-2) = (72+9) - (0+4) = 77$. Again, we don't care about the path, only its endpoints.
16. $\nabla f(x,y,z) =< 2xz + y^2, 2xy, x^2 + 3z^2 >$, so $f(x,y,z) = x^2z + xy^2 + g(y,z) = xy^2 + h(x,z) = x^2y + z^3 + k(x,y)$. We may take $f(x,y,z) = xy^2 + x^2z + z^3$. $r(0) =< 0, 1, -1 >$ and $r(1) =< 1, 2, 1 >$, so the integral is $(4 + 1 + 1) - (-1) = 7$.
17. We get $f(x,y,z) = xy^2 \cos z + g(y,z) = xy^2 \cos z + h(x,z) = xy^2 \cos z + k(x,y)$, so we may take $f(x,y,z) = xy^2 \cos z$. $r(0) =< 0, 0, 0 >$ and $r(\pi) =< \pi^2, 0, \pi >$, so the integral is $0 - 0 = 0$.
18. $f(x,y,z) = xe^y + g(y,z) = xe^y + h(x,z) = ze^z + k(x,y)$, so we may take $f(x,y,z) = xe^y + ze^z$. $r(0) =< 0, 0, 0 >$ and $r(1) =< 1, 1, 1 >$, so we get $2e - 0 = 2e$.
19. To show this is independent of path, check that $Q_x = P_y : Q_x = \sec^2 y = P_y$. Great! To evaluate the integral, we need to find ∇f so that $\int_C \nabla f(x,y) \cdot dr$ is our integral. We have $\nabla f(x,y) =< \tan y, x\sec^2 y >$, so $f(x,y) = x\tan y + g(y) = x\tan y + h(x)$. Taking $f(x,y) = x\tan y$, we get $f(2,\pi/4) - f(1,0) = 2 - 0 = 2$.
20. $Q_x = -e^{-x} = P_y$, so the integral is independent of path. $f(x,y) = x + ye^{-x}$ is a potential function for this, so the integral is $f(1,2) - f(0,1) = (1 + 2e^{-1}) - 1 = \dfrac{2}{e}$.
21. The work done is $\int_C F(x,y) \cdot dr = \int_C 2y^{3/2}dx + 3xy^{1/2}dy$. A quick check shows us that this force field is conservative, so we need a potential function; $f(x,y) = 2xy^{3/2}$ will serve. The work is thus $f(2,4) - f(1,1) = 32 - 2 = 30$ units.
23. This is not conservative. A circle centered at the origin and traversed counterclockwise gives a positive integral since all of the tangents are directed opposite the vector field. However, if the vector field were conservative, then the integral around this closed path would be zero, so it is not conservative.

29. This is the first quadrant; it is open, connected, and simply connected.
30. This is not simply connected; it has a hole in it at the origin. It is, however, both open and connected.
31. This is the annulus, or ring, between circles of radius 1 and 2 centered at the origin. It is open and connected, but it is not simply connected.
32. This is the disk of radius 1 centered at the origin together with the annulus between circles of radius 2 and 3 centered at the origin. It is not closed since it contains its boundary, and it is neither connected nor simply connected.