# Solutions to Homework Assignment 30

**MATH 249-01 and -02**

**Section 16.4 Stewart 6e, Page 1060**

**2, 5-9, 11-14, 17, 19**

2'. [Extra problem: $C$ is the unit circle, $\oint y dx - x dy$.] Parametrize the circle by $r(t) =< \cos t, \sin t >$ on $[0, 2\pi]$. We get $\oint_C y dx - x dy = \int_0^{2\pi} [(\sin t)(-\sin t) - (\cos t)(\cos t)]dt = \int_0^{2\pi} -1 dt = -2\pi$.

$Q_x = -1$ and $P_y = 1$, so by Green's Theorem, we have $\int_D (-1) - (1)dA = \int_0^{2\pi} \int_0^1 (-2) r dr d\theta = \int_0^{2\pi} \frac{-2}{2}[1^2 - 0^2] d\theta = \int_0^{2\pi} -1 d\theta = -2\pi$. I'll be darned! It's the same!

$\int_0^1 \int_{3x}^3 (4y^3 - 2x^2 y) dy dx = \int_0^1 y^4 - x^2 y^2 \Big|_{y=3x}^{y=3} = \int_0^1 [(81 - 9x^2) - (81x^4 - 9x^4)] dx = \int_0^1 81 - 9x^2 - 72x^4 \, dx = 81x - 3x^3$

$\frac{318}{5}$.

7. The parabolas meet at $x = 0$ and $x = 1$; on this interval, $y$ ranges from $x^2$ to $\sqrt{x}$. $Q_x = 2$ and $P_y = 1$, so we get $\int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 1) dy dx = \int_0^1 \sqrt{x} - x^2 dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

8. This looks like a job for Polar Coordinates! $Q_x = 4x^3 + 4xy^2$ and $P_y = 0$, so we get $\int_0^{2\pi} \int_1^2 [4r^3 \cos^3\theta + 4r^3 \cos\theta \sin^2\theta] r dr d\theta = \int_0^{2\pi} \int_0^1 4r^4 \cos\theta dr d\theta = \int_0^{2\pi} \cos\theta d\theta \int_0^1 4r^4 dr = 0$ since integrating $\cos\theta$ over its period gives 0.

9. More polar coordinates! $Q_x = -3x^2, P_y = 3y^2$, so we get $\int_0^{2\pi} \int_0^2 -3(x^2 + y^2) r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 -3r^3 dr = 2\pi\left(-\frac{3}{4}2^4\right) = -24\pi$.

11. The given orientation of $C$ is negative, so we will need a minus sign. $Q_x = 2x$ and $P_y = 3y^2$, so we get $-\int_0^{\pi} \int_0^{\sin x} (2x - 3y^2) dy dx = -\int_0^{\pi} 2x \sin x - \sin^3 x dx = -\int_0^{\pi} 2x \sin x - \sin x(1 - \cos^2 x) dx$. We can integrate the first term by parts ($u = x, dv = \sin x$) and the rest directly or by substitution. We get $-[2(-x \cos x + \sin x) + \cos x - \frac{1}{3}\cos^3 x]$. Evaluation from 0 to $\pi$ gives $-2\pi + 1 + \frac{1}{3} = \frac{4}{3} - 2\pi$.

12. This is also oriented negatively, so we will need a minus sign. $x$ ranges from 0 to 2 and $y$ from 0 to $3x$. $Q_x = 2x + 2y \cos x$ and $P_y = 2y \cos x$, so we get $-\int_0^2 \int_0^{3x} 2x dy dx = -\int_0^2 6x^2 dx = -16$.

13. Clockwise gives a negative orientation. $Q_x = -y^2, P_y = x^2$. We get $-\int_0^{2\pi} \int_0^5 (-y^2 - x^2) r dr d\theta = \int_0^{2\pi} d\theta \int_0^5 r^3 dr = 2\pi \frac{625}{4} = \frac{625\pi}{2}$.

14. Finally, a positive orientation! $Q_x = \frac{2}{1 + (y/x)^2}(-y/x^2) = \frac{-2y}{x^2 + y^2}$ and $P_y = 1 - \frac{2y}{x^2 + y^2}$. We get $\iint_D -1 dA$. This just gives the negative of the area of the circle, which is $\pi$, so the integral is $-\pi$. How lucky is that?

Just for practice, here is an alternative: Let $u = x - 2, v = y - 3$. The curve becomes $u^2 + v^2 = 1$, and $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. The double integral is now $\iint_U (-1)[1] du dv = \int_0^{2\pi} \int_0^1 -1 r dr d\theta = -2\pi(1/2) = -\pi$, where $U$ is the disk of radius 1 centered at the origin.

17. The given curve is oriented positively. We have $Q_x = y^2$ and $P_y = x$. The region is bounded by the $x$- and $y$-axes and the line $y = 1 - x$. The work done is $W = \oint_C F \cdot dr = \iint_D (y^2 - x) dA =$

$$\int_0^1 \int_0^{1-x} (y^2 - x)dydx = \int_0^1 \frac{1}{3}(1-x)^3 - x(1-x)dx = -\frac{1}{12}(0-1) - \frac{1}{2} + \frac{1}{3} = -\frac{1}{12}.$$

19. The width of an arch of the cycloid is $2\pi$. As $t$ increases from 0 to $2\pi$, we travel along the top of the cycloid. To close the path, we then must travel left along the $x$-axis from $2\pi$ to 0. This is a negative orientation, so we will need a minus sign; I will use the second formula. This has $dx = x'(t)dt = 1 - \cos t$, so we get $\int_0^{2\pi} (1-\cos t)(1-\cos t)dt = \int_0^{2\pi} (1-2\cos t+\cos^2 t)dt = \int_0^{2\pi} 1+\frac{1+\cos 2t}{2}dt = (2\pi+\pi) = 3\pi$. The second formula has a minus sign, but we also had a negative orientation, so the two cancelled. Also notice that I dropped the $\cos t$ and $\cos 2t$ from the integral; both of these integrate to 0 on $[0, 2\pi]$.