Solutions to Homework Assignment 33

MATH 249

Section 16.7 Stewart 6e, Page 1091

5, 7, 11, 19, 21, 23

5. For the plane $z = 1 + 2x + 3y$, $dS = \sqrt{2^2 + 3^2 + 1}dA = \sqrt{14}dA$. The integral is $\int_0^3 \int_0^2 x^2y(1 + 2x + 3y)\sqrt{14}dydx = \sqrt{14}\int_0^3 \frac{x^2y^2}{2} + x^3y^2 + x^2y^3\Big|_{y=0}^{y=2} dx = \sqrt{14}\int_0^3 10x^2 + 4x^3\ dx = \sqrt{14}\left(\frac{10}{3}x^3 + x^4\right)\Big|_0^3 = 171\sqrt{14}$.

7. The plane is $z = 1 - x - y$, so $dS = \sqrt{1 + 1 + 1}dA = \sqrt{3}dA$. The part in the first octant lies above the triangle bounded by the x- and y-axes and the line $y = 1 - x$. We have $\int_0^1 \int_0^{1-x} y(1 - x - y)\sqrt{3}dydx = \int_0^1 \frac{1}{2}(1 - x)^2 - \frac{1}{2}x(1 - x)^2 - \frac{1}{3}(1 - x)^3 dx = \frac{\sqrt{3}}{4}$.

11. This portion of the cone lies above the annulus between the circles $r = 1$ and $r = 3$, suggesting that we might want to work in polar coordinates. $z = \sqrt{x^2 + y^2}$, so $dS = \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1}dA = \sqrt{2}dA$. The integral is thus $\iint_S x^2z^2 dS = \sqrt{2}\int_0^{2\pi}\int_1^3 r^2\cos^2\theta(r^2)rdrd\theta = \sqrt{2}\int_0^{2\pi}\int_1^3 r^5\frac{1 + \cos 2\theta}{2}drd\theta = \sqrt{2}\int_0^{2\pi}\frac{1}{12}(3^6 - 1)d\theta = \frac{728\sqrt{2}}{12}(2\pi) = \frac{364\sqrt{2}\pi}{3}$. (Note that $\int_0^{2\pi}\cos 2\theta d\theta = 0$, so I dropped that term.)

19. We are above the square, so our limits for x and y are 0 to 1 in both cases. The function g describing the surface is $g(x, y) = 4 - x^2 - y^2$. We have $\int_0^1\int_0^1 -Pg_x - Qg_y + Rdydx = \int_0^1\int_0^1 [(-xy)(-2x) - y(4 - x^2 - y^2)(-2y) + x(4 - x^2 - y^2)]dydx = \int_0^1\int_0^1 [2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2]dydx = \frac{713}{180}$.

21. With a downward orientation, we compute the flux with the integral $\iint_D Pg_x + Qg_y - RdA$. Here, $z = 1 - x - y$, so $g_x = g_y = -1$. This meets the xy-plane, $z = 0$, in the line $x + y = 1$. We get $\int_0^1\int_0^{1-x}[(-xze^y) - (xze^y)(-1) - z]dydx = \int_0^1\int_0^{1-x}(x + y - 1)dydx = -\frac{1}{6}$.

23. In the first octant, the surface is given by $z = g(x, y) = \sqrt{4 - x^2 - y^2}$. Thus $g_x = -\frac{x}{4 - x^2 - y^2}$ and $g_y = -\frac{y}{4 - x^2 - y^2}$. Orientation toward the origin implies that we need to use $\iint_D Pg_x + Qg_y - RdA = \iint_D \frac{-x^2}{\sqrt{4 - x^2 - y^2}} + \frac{y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}} - ydA$. In polar coordinates, this becomes $\int_0^{\pi/2}\int_0^2 \frac{-r^3\cos^2\theta}{\sqrt{4 - r^2}}drd\theta = -\left(\int_0^{\pi/2}\frac{1 + \cos 2\theta}{2}d\theta\right)\left(\int_0^2 \frac{r^3}{\sqrt{4 - r^2}}dr\right)$. The first integral is just $\frac{\pi}{4}$. For the second integral, substitute $u = 4 - r^2$, so $du = -2rdr$ and $r^2 = 4 - u$. That integral becomes $\int_4^0 -\frac{1}{2}\frac{4 - u}{u^{1/2}}du = \frac{1}{2}\int_0^4 4u^{-1/2} - u^{1/2}\ du = 4u^{1/2} - \frac{1}{3}u^{3/2}\Big|_0^4 = 8 - \frac{8}{3} = \frac{16}{3}$. The product is $-\frac{4\pi}{3}$.