

Solutions to Homework Assignment 33

MATH 249

Section 16.8 Stewart 6e, Page 1097

1, 3-10, 13, 15

- The two surfaces are both orientable and they have the same boundary curve, which is the circle $x^2 + y^2 = 4$ in the plane. If we give them both a positive orientation, then both are equal to $\int_C F \cdot dr$, and therefore equal to each other.
- $F = \langle x^2y^3z, \sin xyz, xyz \rangle$. The circle $x^2 + z^2 = 9$ in the plane $y = 3$ bounds this cone. If we stand so our head is in the direction of the y -axis, we must proceed counterclockwise to keep the surface on our left. We can therefore parametrize the circle by $r(t) = \langle 3 \cos t, 3, -3 \sin t \rangle$, so $r'(t) = \langle -3 \sin t, 0, -3 \cos t \rangle$.

$$\begin{aligned}
 \iint_S \text{curl}F \cdot d\vec{S} &= \int_C \langle x^2y^3z, \sin xyz, xyz \rangle \cdot \langle -3 \sin t, 0, -3 \cos t \rangle dt \\
 &= \int_0^{2\pi} [(9 \cos^2 t)(3^3)(-3 \sin t)(-3 \sin t) + 0 + (3 \cos t)(3)(-3 \sin t)(-3 \cos t)] dt \\
 &= \int_0^{2\pi} [2187 \sin^2 t \cos^2 t + 81 \cos^2 t \sin t] dt \\
 &= \int_0^{2\pi} 2187 \frac{\sin^2 2t}{4} dt \\
 &= \int_0^{2\pi} \frac{2187}{8} (1 - \cos 4t) dt \\
 &= \frac{2187}{8} (2\pi) \\
 &= \frac{2187\pi}{4}.
 \end{aligned}$$

- Instead of integrating over the given surface, let's integrate over the base. Both surfaces are bounded by the same square, so we will get the same result. The boundary square needs to be parametrized in four pieces. We are to be oriented outward, which means downward for the bottom face. We will need to traverse in the direction from $(1, 1, -1)$ to $(1, -1, -1)$. The four boundary curves are parametrized by $r(t) = \langle 1, t, -1 \rangle$ from $t = 1$ to $t = -1$, $r(t) = \langle t, -1, -1 \rangle$ from $t = 1$ to $t = -1$, $r(t) = \langle -1, t, -1 \rangle$ from $t = -1$ to $t = 1$, and $r(t) = \langle t, 1, -1 \rangle$ from $t = -1$ to $t = 1$. Traversing these in this order will move us around the square with a positive orientation. Now

$$\begin{aligned}
 \iint_S \text{curl}F \cdot d\vec{S} &= \int_C \langle xyz, xy, x^2yz \rangle \cdot dr \\
 &= \int_1^{-1} \langle xyz, xy, x^2yz \rangle \cdot \langle 0, 1, 0 \rangle dt + \int_1^{-1} \langle xyz, xy, x^2yz \rangle \cdot \langle 1, 0, 0 \rangle dt \\
 &+ \int_{-1}^1 \langle xyz, xy, x^2yz \rangle \cdot \langle 0, 1, 0 \rangle dt + \int_{-1}^1 \langle xyz, xy, x^2yz \rangle \cdot \langle 1, 0, 0 \rangle dt \\
 &= \int_1^{-1} xy dt + \int_1^{-1} xyz dt + \int_{-1}^1 (xy) dt + \int_{-1}^1 xyz dt \\
 &= 0.
 \end{aligned}$$

- $F = \langle e^{xy} \cos z, x^2z, xy \rangle$. To traverse the boundary in a positive orientation, we take $r(t) = \langle 0, \cos t, \sin t \rangle$, so $r'(t) = \langle 0, -\sin t, \cos t \rangle$. On this curve, $F = \langle \cos z, 0, 0 \rangle$, so $F \cdot r'(t) = 0$. Therefore, $\int_C F \cdot dr = 0$.
- The plane given is $x + y + z = 1$, or $z = 1 - x - y$ and $\text{curl}F = \langle -2z, -2x, -2y \rangle$. Now

$$\begin{aligned}
\int_C &= \iint_S \operatorname{curl} F \cdot d\vec{S} \\
&= \int_0^1 \int_0^{1-x} (-Pg_x - Qg_y + R) dy dx \\
&= \int_0^1 \int_0^{1-x} (2z(-1) + 2x(-1) + (-2y)) dy dx \\
&= \int_0^1 \int_0^{1-x} (-2(1-x-y) - 2x - 2y) dy dx \\
&= \int_0^1 \int_0^{1-x} -2 dy dx \\
&= -1.
\end{aligned}$$

8. $F(x, y, z) = \langle e^{-x}, e^x, e^z \rangle$, so $\operatorname{curl} F = \langle 0, 0, e^x \rangle$. $z = g(x, y) = 1 - x - \frac{1}{2}y$. When $z = 0$, $y = 2 - 2x$. We get

$$\begin{aligned}
\int_C &= \iint_S \operatorname{curl} F \cdot d\vec{S} \\
&= \int_0^1 \int_0^{2-2x} (-Pg_x - Qg_y + R) dy dx \\
&= \int_0^1 \int_0^{2-2x} (0 + 0 + e^x) dy dx \\
&= \int_0^1 (2 - 2x)e^x dx \\
&= 2e - 4.
\end{aligned}$$

9. $\operatorname{curl} F = \langle xe^{xy} - 2x, y - ye^{xy}, z \rangle$. The region of integration is the disk $x^2 + y^2 \leq 16$. Notice that $g(x, y) = z = 5$, so $g_x = g_y = 0$. We get

$$\begin{aligned}
\int_C &= \iint_S \operatorname{curl} F \cdot d\vec{S} \\
&= \int_0^{2\pi} \int_0^4 (-Pg_x - Qg_y + R) r dr d\theta \\
&= \int_0^{2\pi} \int_0^4 (0 + 0 + 5) r dr d\theta \\
&= 2\pi \cdot \frac{5}{2} (4^2) \\
&= 80\pi.
\end{aligned}$$