MATH 249

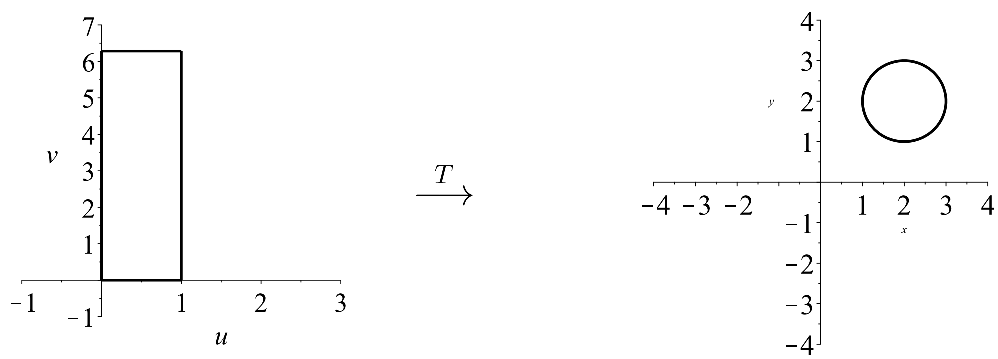
Today

1. 15.9: General Change of Variables (Understand how to change variables in multiple integrals and how to use the Jacobian.)

2. WeBWorK

15.9: General Change of Variables

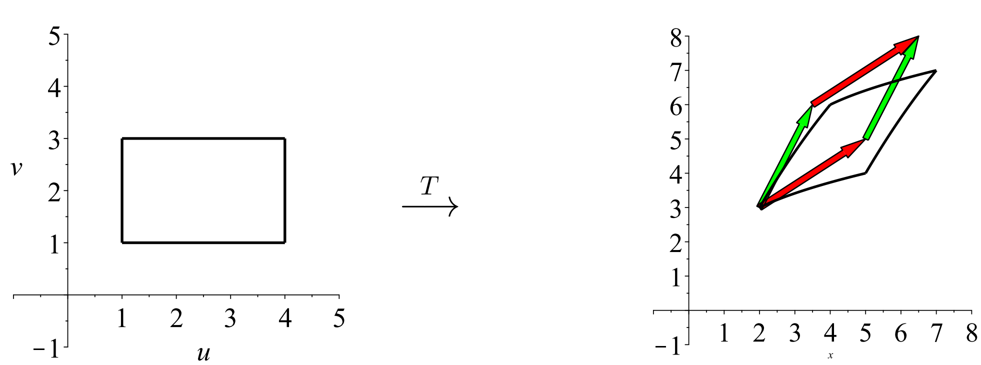
1. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one (so it has an inverse), and assume that T maps the set S in the uv-plane to the set R in the xy-plane: $T(u, v) = (x, y)$ and $T^{-1}(x, y) = (u, v)$.



2. Now think small: imagine that S is a "nice" region in the uv-plane (e.g., a rectangle) and assume that we have subdivided it into little subrectangles as is our custom.

3. Let $\vec{r}(u, v) =< g(u, v), h(u, v) >=< x, y >$ be the position vector of $T(u, v)$. Note that $\vec{r}$ parametrizes the boundary of S. Also, for purposes of taking the cross product later, we will consider $\vec{r}$ as a three-dimensional vector $< x, y, 0 >$.

4. Consider one subrectangle S_{ij} of dimensions Δu and Δv:



5. T maps S_{ij} into $\mathbb{R}^2$. If we use tangents to the sides as approximations for the sides themselves (legitimate if T is continuous and Δu and Δv are small), then the area of $T(S_{ij})$ is approximately

$$\Delta A = |(\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v)|$$

(remember?!?), or

$$\pm \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \Delta u \Delta v = \pm \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v.$$

(We need to choose the sign so that the area is positive.)

6. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ is called the **Jacobian** of T. The vertical bars refer to the **determinant**, not absolute value.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation with continuous partial derivatives and having a nonzero Jacobian. Let S be a Type I or Type II region in the uv-plane, and let $R = T(S)$ also be Type I or Type II. Assume that T is one-to-one except perhaps on the boundary of S. If f is continuous on R, then

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

Note: this uses the absolute value of the Jacobian.

8. Example: polar coordinates.

9. Examples p. 1020: #2, 3, 9, 10, 11, 13, 19, 22

Next Time

1. Watch 16.1 [~17 minutes], start 16.2 [first ~20 minutes]

2. Homefun/Python