MATH 249

Today

1. 16.7: Surface Integrals (Understand the definition, calculation, interpretation, and application of surface integrals)
2. WeBWorK
3. Homefun

16.7: Surface Integrals

1. Suppose we have a **surface density** on a surface S given by $f(x, y, z)$ – it could be density of charge, mass, population, etc. – and we want to add it up over the surface. Also assume that the surface is described parametrically by $\vec{r}(u, v)$ over a region D in the uv-plane.

2. Subdivide D in the usual way, and choose a sample point P_{ij}^* from each subrectangle. The mass (say) of the corresponding patch on S is approximately $f(P_{ij}^*)\Delta S_{ij}$, where ΔS_{ij} is the area of that patch. Adding these up over D gives an approximation to the total mass of S. But we don't want an approximation, we want an exact value! Accordingly, we define the **surface integral** of f over S by

$$\iint_S f(x, y, z)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

3. As usual, our task is to determine the differential dS. From the previous section, we know that $\Delta S_{ij} \approx |\vec{r}_u(P_{ij}^*) \times \vec{r}_v(P_{ij}^*)|\Delta A$, so we get

$$\iint_S f(x, y, z)dS = \iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|dA,$$

integrating over D in the uv-plane.

4. For this integral to be defined, we need $\vec{r}_u$ and $\vec{r}_v$ to be continuous, nonzero, and nonparallel (i.e., S is smooth).

5. Note that $A(S) = \iint_S 1dS$.

6. If $S : z = g(x, y)$, then this becomes

$$\iint_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{g_x^2 + g_y^2 + 1}dA.$$

7. An **orientable** surface S is one with a normal that varies continuously over S (like a sphere or a torus, for example).

8. If $S : z = g(x, y)$, then $\hat{n} = \dfrac{< -g_x, -g_y, 1 >}{\sqrt{g_x^2 + g_y^2 + 1}}$ gives the upward unit normal to S. (That's a choice in orientation.)

9. If $S : r(u, v)$ is a smooth, orientable parametric surface, then $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ gives a unit normal. (Which direction depends on the choice of orientation.)

10. If $\vec{F} =< P, Q, R >$ is a continuous vector field defined on an oriented surface S with unit normal $\hat{n}$, we define the **surface integral of $\vec{F}$ over** S by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS.$$

11. Note that $\vec{F} \cdot \hat{n}$ gives the projection of $\vec{F}$ through S as in our modified approach to Green's Theorem.

12. This integral is called a **flux** integral.

13. For a parametric surface, we get

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA.$$

14. For $S : z = g(x, y)$, we get

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R) dA.$$

15. Examples p. 1091: #5, 9, 19, 22, 28

16. Examples : Taalman p. 1120, #41

17. WeBWorK: #3, 8

Next Time

1. Watch 16.8 [∼45 minutes]