Math 251-02/03 Homework 10

April 15, 2012

Rough drafts due 4/16/12, edits due 4/18/12, final drafts due 4/20/12.

Problems to Keep:

1. **6.3.1 (2,3)** Prove the formulas hold for all $n \in \mathbb{N}$.

   (2) $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   Base Case: If $n = 1$, the left-hand side is 1 and the right-hand side is also 1, so the Base Case holds.

   Induction Step: Now assume that for some $n \in \mathbb{N}$, $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$. We want to show that $1^2+2^2+\ldots+n^2+(n+1)^2 = \dfrac{(n+1)(n+2)(2n+3)}{6}$. We calculate:

$$
\begin{aligned}
1^2 + 2^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right) \\
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= (n+1)\left(\frac{2n^2+n+6n+6)}{6}\right) \\
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= (n+1)\left(\frac{2n^2+7n+6)}{6}\right) \\
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= (n+1)\left(\frac{(n+2)(2n+3))}{6}\right) \\
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= \frac{(n+1)(n+2)(2n+3)}{6},
\end{aligned}
$$

   as desired. Therefore, by the Principle of Mathematical Induction, $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

   (3) $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

   Base Case: If $n = 1$, the left-hand side is 1 and the right-hand side is also 1, so the Base Case holds.

Induction Step: Now assume that for some $n \in \mathbb{N}$, $1^3+2^3+\ldots+n^3 = \dfrac{n^2(n+1)^2}{4}$. We want to show that $1^3+2^3+\ldots+n^3+(n+1)^3 = \dfrac{(n+1)^2(n+2)^2}{4}$. We calculate:

$$\begin{aligned}
1^3+2^3+\ldots+n^3 &= \frac{n^2(n+1)^2}{4} \\
1^3+2^3+\ldots+n^3+(n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
1^3+2^3+\ldots+n^3+(n+1)^3 &= (n+1)^2\left(\frac{n^2}{4}+n+1\right) \\
1^3+2^3+\ldots+n^3+(n+1)^3 &= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\
1^3+2^3+\ldots+n^3+(n+1)^3 &= (n+1)^2\left(\frac{(n+2)^2}{4}\right) \\
1^3+2^3+\ldots+n^3+(n+1)^3 &= \frac{(n+1)^2(n+2)^2}{4},
\end{aligned}$$

as desired. Therefore, by the Principle of Mathematical Induction, $1^3+2^3+\ldots+n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

2. **6.3.2** Prove that $1+2n \leq 3^n$ for all $n \in \mathbb{N}$.

   Base Case: If $n=1$, then $1+2n = 1+2(1) = 3 \leq 3^1 = 3^n$, so the base case holds.

   Induction Step: Assume that for some $n \geq 1$, $1+2n \leq 3^n$. We want to show that $1+2(n+1) \leq 3^{n+1}$, which is to say that $3+2n \leq 3^{n+1}$.

   Since $1+2n \leq 3^n$, we know that $3+2n \leq 3^n+2$. But $2 \leq 2\cdot 3^n$ since $n \geq 1$, so $3+2n \leq 3^n + 2\cdot 3^n = 3^{n+1}$, as desired.

   Therefore, by the Principle of Mathematical Induction, $1+2n \leq 3^n$ for all $n \in \mathbb{N}$..

3. **6.3.6** For which values of n does $n^2-9n+19>0$ hold?

   Completing the square gives $n^2-9n+19 = n^2-9n+20.25-1.25 = (n-4.5)^2-1.25$. For this to be greater than 0, we need $n \geq 6$ or $n \leq 3$. It is easy to check that the inequality holds for $n = 1,2,3$ and fails for $n = 4,5$.

   Base Case: $n=6$: The inequality is easily seen to hold, so the base case is established.

   Induction Step: Assume that for some $n \in N$, $n^2-9n+19>0$. We want to show that $(n+1)^2-9(n+1)+19>0$. Compute:

$$\begin{aligned}
(n+1)^2-9(n+1)+19 &= (n^2+2n+1)+(-9n-9)+19 \\
&= (n^2-9n+19)+(2n-8) \\
&> 2n-8 \\
&\geq 0
\end{aligned}$$

   since $n \geq 6$ implies $2n-8 \geq 0$. Note the use of the Induction Hypothesis in going from line 2 to line 3.

Therefore $(n+1)^2 - 9(n+1) + 19 > 0$, so by the Principle of Mathematical Induction, $n^2 - 9n + 19 > 0$ for all $n \geq 6$.

4. **6.3.12** Prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$ for all $n \geq 2$.

   Base Case: If $n = 2$, the left-hand side is $1 + \frac{1}{2} = 3/2$, while the right-hand side is $\sqrt{2} \approx 1.414$, so the inequality holds. This establishes the base case.

   Induction Step: Now assume that for some $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$. We want to show that $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}$.

   We calculate:

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} &= \left(\sum_{i=1}^{n} \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{n+1}} \\
&> \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&= \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} \\
&> \frac{n+1}{\sqrt{n+1}} \\
&= \sqrt{n+1}.
\end{aligned}$$

   Note that $\sqrt{n}\sqrt{n+1} > n$ since $\sqrt{n+1} > \sqrt{n}$.

   Therefore $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}$, so by the Principle of Mathematical Induction, $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$ for all $n \in \mathbb{N}$.