Math 251-02/03 Homework 12

May 8, 2012

Problems to Turn in:
Problems to Keep:

1. **6.5.1** Prove that $\mathbb{Z} \sim 5\mathbb{Z}$.

   Define $f : \mathbb{Z} \to 5\mathbb{Z}$ by $f(n) = 5n$. If $f(m) = f(n)$, then $5m = 5n$, so $m = n$. Thus f is injective. Since now $\mathbb{Z} \preceq 5\mathbb{Z} \preceq \mathbb{Z}$, the Schroeder-Bernstein Theorem implies that $\mathbb{Z} \sim 5\mathbb{Z}$.

2. **6.5.2** Prove that the disk D of radius 3 centered at $(1, 2)$ has the same cardinality as U, the unit disk centered at the origin.

   Define $f : D \to U$ by $f(x, y) = 1/3(x - 1, y - 2)$. If $f(x, y) = f(z, w)$, then $1/3(x - 1, y - 2) = 1/3(z-1, w-2)$, so $x-1 = z-1$ and $y-2 = w-2$. Therefore, $(x, y) = (z, w)$, so f is injective. Now let $(x, y) \in U$. Then $f(3x+1, 3y+2) = 1/3((3x+1)-1, (3y+2)-2) = 1/3(3x, 3y) = (x, y)$, so f is surjective as long as we can verify that $(3x + 1, 3y + 2) \in D$. To do so, we find the distance between that point and $(1, 2)$: $\sqrt{(3x + 1 - 1)^2 + (3x + 2 - 2)^2} = 3\sqrt{x^2 + y^2} \leq 3$ since $(x, y) \in U$ implies that $x^2 + y^2 \leq 1$.

   Since there is a bijection (namely, f) between U and D, they have the same cardinality.

3. **6.5.6**

   (1) Let $A = \{1, 2\}, B = \{1, 3\}$, and $\mathbb{C} = \{3, 4\}$. Then $A \cup C = \{1, 2, 3, 4\}$, while $B \cup C = \{1, 3, 4\}$, so the two sets have different cardinalities.

   (2) Since $A \sim B$, there is a bijection $f : A \to B$. Define $g : A \cup C \to B \cup C$ by

   $$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in C \end{cases}.$$

   Note that g is well defined since $A \cap C = \emptyset$. If $g(x) = g(y)$, then, since $B \cap C = \emptyset$, either $g(x)$ and $g(y)$ are in B (in which case x and y are both in A), or they are both in C (in which case x and y are both in C). In the first case, we have $f(x) = g(x) = g(y) = f(y)$, so $x = y$ by the injectivity of f. In the second case, we have $x = g(x) = g(y) = y$, so again $x = y$. In either case, we find that g is injective.

   Now let $y \in B \cup C$. If $y \in B$, then, by the surjectivity of f, there is some $x \in A$ such that $f(x) = y$ and therefore $g(x) = y$. If $y \in C$, then $g(y) = y$. Thus, in either case, there is an element $x \in A \cup C$ such that $g(x) = y$, so g is surjective.

   (3) This is false. Let $A = \{0\}, B = \{-1, 0\}$, and $C = \mathbb{N}$. Then $A \cup C$ and $B \cup C$ are both countably infinite, but A and B are not equicardinal.

4. **6.5.9** $A \times \{x\} \sim A$.

   Define $f : A \times \{x\} \to A$ by $f(a, x) = a$ for all $a \in A$. If $f(a, x) = f(b, x)$, then $a = b$, so f is injective. If $a \in A$, then $f(a, x) = a$, so f is surjective. Therefore, $A \times \{x\} \sim A$. (If A is empty, this is clear.)

5. **6.5.11**

   (1) The empty function is injective.

   (2) The identity map is injective.

   (3) The composition of injective functions is injective.

6. **6.6.1** Suppose that A and B are finite sets. Prove that $A \cup B$ is finite.

   If either set is empty, the union is equal to the other and therefore finite. Assume now that A and B are finite and nonempty. Since A and B are finite, there exist bijections $f : A \to \{1, \ldots, m\}$ and $g : B \to \{1, \ldots, n\}$ for some $m, n \in \mathbb{N}$. Define $h : A \cup B \to \{1, \ldots, m+n\}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) + m & \text{if } x \in B - A \end{cases} .$$

   Note that h is injective, so $A \cup B \preceq \{1, \ldots, m+n\}$, so $A \cup B \preceq \{1, \ldots, r\}$ for some $r \in \mathbb{N}$. Therefore, $A \cup B$ is finite.

7. **6.6.4** (This is a good one to know.) Let A and B be finite sets with $|A| = |B|$, and let $f : A \to B$ be a function. Prove that f is bijective if and only if f is injective if and only if f is surjective.

   We first prove a lemma: a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is bijective if and only if f is injective if and only if f is surjective.

   If f is bijective, then it is injective. Suppose that f is injective. By Lemma 6.3.11 (2), there is a bijection $g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $g(f(\{1, \ldots, n\}) = \{1, \ldots, r\}$ for some $r \le n$. Thus $\{1, \ldots, n\} \preceq f(\{1, \ldots, n\}) \preceq \{1, \ldots, r\} \preceq \{1, \ldots, n\}$. Therefore, by the Schroeder-Bernstein Theorem, $\{1, \ldots, r\} \sim \{1, \ldots, n\}$. By Lemma 6.6.2, this implies that $r = n$. Now since g is a bijection, we know that $f(\{1, \ldots, n\}) \sim g \circ f(\{1, \ldots, n\}) \sim \{1, \ldots, n\}$. By Theorem 6.6.5 (4), $f(\{1, \ldots, n\})$ cannot be a proper subset of $\{1, \ldots, n\}$, so we must have $f(\{1, \ldots, n\}) = \{1, \ldots, n\}$, proving that f is surjective.

   Next, assume that f is surjective. We have $f(\{1, \ldots, n\}) = \{1, \ldots, n\}$. Suppose that $f(a) = f(b)$ for some $a, b \in \{1, \ldots, n\}$ with $a \neq b$. Then $f : \{1, \ldots, n\} - \{b\} \to \{1, \ldots, n\}$ is still surjective. This is not possible: there would then be a *bijection* from $\{1, \ldots, n\} - \{b\}$ to $\{1, \ldots, n\}$, which is not possible by Theorem 6.6.5(4). Therefore, f is also injective, proving the lemma.

   Now we return to sets A and B. Let $n = |A| = |B|$; that is, assume that there are bijections $g : \{1, \ldots, n\} \to A$ and $h : \{1, \ldots, n\} \to B$.

   If f is bijective, then f is certainly injective, so assume f is injective. Then $h^{-1} \circ f \circ g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is injective and therefore bijective by the lemma. Since h and g are bijective, $h \circ (h^{-1} \circ f \circ g) \circ g^{-1} = f$ is bijective as well.

   Now assume f is surjective. Then Then $h^{-1} \circ f \circ g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is surjective and therefore bijective by the lemma. Since h and g are bijective, $h \circ (h^{-1} \circ f \circ g) \circ g^{-1} = f$ is bijective as well.