Math 251-02/03 Solutions Homework 9 Suggested Problems

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Rough drafts due 4/9, edits due 4/11, final drafts due 4/13.

Problems to Keep:

1. **5.1.1(1-3)** For each of the following relations on $\mathbb{Z}$, find the relation classes $[3], [-3]$, and $[6]$.

   (1) $S : aSb$ if and only if $a = |b|$ on $\mathbb{Z}$.
   $[3] = \{x \in \mathbb{Z} | 3Sx\} = \{x \in \mathbb{Z} | 3 = |x|\} = \{\pm 3\}$.
   $[-3] = \{x \in \mathbb{Z} | -3Sx\} = \{x \in \mathbb{Z} | -3 = |x|\} = \emptyset$.
   $[6] = \{x \in \mathbb{Z} | 6Sx\} = \{x \in \mathbb{Z} | 6 = |x|\} = \{\pm 6\}$.

   (2) $D : aDb$ if and only if $a|b$ on $\mathbb{Z}$.
   $[3] = \{x \in \mathbb{Z} | 3Dx\} = \{x \in \mathbb{Z} : 3|x\} = \{0, \pm 3, \pm 6, \ldots\}$.
   $[-3] = \{x \in \mathbb{Z} | -3Dx\} = \{x \in \mathbb{Z} : -3|x\} = \{0, \pm 3, \pm 6, \ldots\} = [3]$.
   $[6] = \{x \in \mathbb{Z} | 3Dx\} = \{x \in \mathbb{Z} : 3|x\} = \{0, \pm 6, \pm 12, \ldots\}$.

   (3) $T : aTb$ if and only if $b|a$ on $\mathbb{Z}$.
   $[3] = \{x \in \mathbb{Z} | 3Tx\} = \{x \in \mathbb{Z} : x|3\} = \{\pm 1, \pm 3\}$.
   $[-3] = \{x \in \mathbb{Z} | -3Tx\} = \{x \in \mathbb{Z} : x| -3\} = \{\pm 1, \pm 3\}$.
   $[6] = \{x \in \mathbb{Z} | 6Tx\} = \{x \in \mathbb{Z} : x|6\} = \{\pm 1, \pm 2, \pm 3, \pm 6\}$.

2. **5.1.3(2, 4, 5, 6)** Let $A = \{1, 2, 3\}$. Each of the following subsets of $A \times A$ defines a relation on A. Is each relation reflexive, symmetric, and/or transitive?

   (2) $\overline{N} = \{(1,1), (2,2), (3,3), (1,2)\}$. $\overline{N}$ is reflexive and transitive, but not symmetric (since $(2,1) \notin \overline{N}$).

   (4) $\overline{P} = \{(1,1), (2,2), (3,3)\}$. $\overline{P}$ is reflexive, symmetric, and transitive. (This is equality.)

   (5) $\overline{Q} = \{(1,2), (2,1), (1,3), (3,1), (1,1)\}$. $\overline{Q}$ is not reflexive, but it is symmetric and transitive.

   (6) $\overline{R} = \{(1,2), (2,3), (3,1)\}$. $\overline{R}$ is not reflexive, symmetric, or transitive. ($1R2$ and $2R3$, but $1 \not R 3$.)

3. **5.1.6** Find the flaw in the argument that a symmetric and transitive relation is reflexive.

   The problem is that there may not be a y such that xRy.

4. **5.2.1** True or False.

   (a) $13 \equiv 5 \pmod{2}$. True: $2|13-5$.

   (b) $21 \equiv 7 \pmod{5}$. False: $5 \not| 21-7$.

   (c) $7 \equiv 7 \pmod{2}$. False: $2 \not| 7-0$.

   (d) $3 \equiv 28 \pmod{5}$. True: $5|28-3$.

   (e) $23 \equiv 23 \pmod{7}$. True: $7|23-23$.

5. **5.2.2** Solve modulo n. In each case, let $x = [a]$ (i.e., let a be a representative for x modulo n).

   (a) $[5] + x = [1]$ in $\mathbb{Z}_9$:

$$\begin{aligned}[5] + x &= [1]\\ [5] + [a] &= [1]\\ [5 + a] &= [1]\end{aligned}$$

   Thus $5 + a \equiv 1 \pmod{9}$, so $a \equiv -4 \equiv 5 \pmod{9}$. Therefore, $x = [5]$. Check: $[5] + [5] = [10] = [1]$.

   (b) $[2]x = [7]$ in $\mathbb{Z}_{11}$.

$$\begin{aligned}[2]x &= [7]\\ [2][a] &= [7]\\ [6]([2][a]) &= [6][7]\\ ([6][2])[a] &= [42]\\ [12][a] &= [9]\\ [1][a] &= [9]\\ [1 \cdot a] &= [9]\\ [a] &= [9].\end{aligned}$$

   Check: $[2][9] = [18] = [7]$.

   (c) $x[6] = [4]$ in $\mathbb{Z}_{15}$.
   In this case, we must have $15|6x - 4$, so, in particular, we must have $3|6x - 4$. Thus $6x - 4 = 3k$ for some $k \in \mathbb{Z}$. But then $4 = 6x - 3k = 3(2x - k)$, so $3|4$, which is a contradiction. Therefore, this has no solution in $\mathbb{Z}_{15}$.

   (d) $x[6] = [2]$ in $\mathbb{Z}_{10}$. A quick check reveals that $x = [2]$ is a solution.

   (e) $[3]x + [4] = [1]$ in $\mathbb{Z}_5$. A little work (similar to what is above) yields $x = [4]$.

6. **5.3.1(2, 4, 5, 6)**

(2) $S : xSy$ if and only if $x = |y|$ is not an equivalence relation since $-3 \not S -3$ (S is not reflexive).

(4) $Z : xZy$ if and only if x and y are first cousins is not reflexive and hence not an equivalence relation.

(5) $R : xRy$ if and only if x and y have the same maternal grandmother. This is reflexive; each person has the same maternal grandmother as himself or herself. It is also symmetric since if x and y have the same maternal grandmother, so do y and x. Finally, it is transitive since if x and y have the same maternal grandmother and y and z have the same maternal grandmother, so do x and z.

(6) $W : xWy$ if and only if $x||y$. This depends on whether one considers a line to be parallel to itself. (Some people do, and some people do not.) I do, so, for me, this is an equivalence relation. (The only question is with the reflexive property: parallelism is symmetric and transitive regardless of whether a line is to be considered parallel to itself.)

7. **5.3.2(2,3)** Find [0] and [3] for each relation.

(2) $S : aSb$ if and only if $\sin a = \sin b$. $[0] = \{\pi n | n \in \mathbb{Z}\}$ and $[3] = \{3 + 2\pi n | n \in \mathbb{Z}\} \cup \{\pi - 3 + 2\pi n | n \in \mathbb{Z}\}$. (Note that $\sin x = \sin(\pi - x)$ and $\sin x$ has period 2π.)

(3) $T : aTb$ if and only if there is some $n \in \mathbb{Z}$ such that $a = 2^n b$.
$[0] = \{b \in \mathbb{Z} | 0Tb\} = \{b \in \mathbb{Z} | 0 = 2^n b \text{ for some } n \in \mathbb{Z}\} = \{0\}$.
$[3] = \{b \in \mathbb{N} | 3Tb\} = \{b \in \mathbb{Z} | 3 = 2^n b \text{ for some } n \in \mathbb{Z}\} = \{1, 3, 6, 12, 24, \ldots\} = \{3 \cdot 2^m | m \in \mathbb{Z}, m \leq 0\}$.

8. **5.3.3** Give a geometric description of each equivalence class.

(a) $Q : (x, y)Q(z, w)$ if and only if $x^2 + y^2 = z^2 + w^2$. Since $x^2 + y^2$ is the distance of the point (x, y) from the origin, each equivalence class is made up of points equidistant from the origin. That is, equivalence classes are circles centered at the origin.

(b) $U : (x, y)U(z, w)$ if and only if $|x| + |y| = |z| + |w|$. The equation $|x| + |y| = C$ produces a square center at the origin and rotated 45 degrees relative to the axes so that the vertices lie at $(C, 0), (-C, 0), (0, C)$, and $(0, -C)$. These squares are the equivalence classes. (These are "circles" in what is called the "taxicab metric.")

(c) $V : (x, y)V(z, w)$ if and only if $\max\{|x|, |y|\} = \max\{|z|, |w|\}$.

The lines $y = \pm x$ serve as boundaries for the equivalence classes in some sense since crossing those lines changes whether $|x|$ or $|y|$ is larger. If y lies between $\pm x$, then $\max\{|x|, |y|\} = |x|$; note that this is constant if x is held constant. That means that all such points (x, y) lie in the same equivalence class.

On the other hand, if y is outside of that range (and x is held constant), then $\max\{|x|, |y|\} = |y|$, which is different for every choice of $|y|$ even though x is held constant; these points are in different equivalence classes.

That is, given a, the set $A_a = \{(a, b) : -|a| \leq b \leq |a|\}$ is a subset of $[(a, b)]$. This set is made up of the two vertical line segments joining $y = -x$ and $y = x$ at $x = a$. Note also that $B_b = \{(b, a) : -|a| \leq b \leq |a|\}$ has the same property: if

$(b, a) \in B_b$, then $\max\{|a|, |b|\} = |a|$. Thus B_b is also a subset of $[(a, b)]$, and B_b is made up of two horizontal line segments joining $y = \pm x$ at $y = a$. Together, A_a and B_b form a square centered at the origin that has its sides parallel to the axes. These are the equivalence classes. (If you leave the square, the max changes.)