Solutions to Homework Assignment 2

MATH 256-01
Section 1.2, Page 14
To Keep: 1, 2, 6, 11, 12, 13, 15

1. (a) Since $\dfrac{dy}{dt} = -y + 5$, we have

$$\begin{aligned} \frac{dy/dt}{-y+5} &= 1 \\ -\frac{d}{dt}[\ln|-y+5|] &= 1 \\ -\ln|-y+5| &= t + C \\ \ln|-y+5| &= -t + C \\ |-y+5| &= e^{-t+C} \\ -y+5 &= \pm Ce^{-t} \\ y &= 5 \pm Ce^{-t}. \end{aligned}$$

Notice that I have used the same symbol C throughout to represent the constant, even though the C at the end is actually $-e^{-C}$. (Check it!) I will continue with this practice.
Since $y(0) = 5 \pm C = y_0$, we must have $C = \pm(y_0 - 5)$. Thus, $y = 5 + (y_0 - 5)e^{-t}$.

(b) Here we have $\dfrac{dy}{dt} = -2y + 5$, so

$$\begin{aligned} \frac{dy/dt}{-2y+5} &= 1 \\ -\frac{1}{2}\frac{d}{dt}[\ln|-2y+5|] &= 1 \\ \frac{d}{dt}[\ln|-2y+5|] &= -2 \\ |-2y+5| &= Ce^{-2t} \\ -2y+5 &= \pm Ce^{-2t} \\ y &= \frac{5}{2} \pm Ce^{-2t}. \end{aligned}$$

With $y(0) = y_0$, we have $\dfrac{5}{2} \pm C = y_0$, so $C = \pm(y_0 - 5/2)$. Thus, $y = \dfrac{5}{2} \pm \left(y_0 - \dfrac{5}{2}\right) e^{-2t}$.

(c) $\dfrac{dy}{dt} = -2y + 10$, so

$$\begin{aligned} \frac{dy/dt}{-2y+10} &= 1 \\ -\frac{1}{2}\frac{d}{dt}[\ln|-2y+10|] &= 1 \\ \frac{d}{dt}[\ln|-2y+10|] &= -2 \\ |-2y+10| &= Ce^{-2t} \\ -2y+10 &= \pm Ce^{-2t} \\ y &= 5 \pm Ce^{-2t}. \end{aligned}$$

With $y(0) = y_0$, we have $5 \pm C = y_0$, so $C = \pm(y_0 - 5)$. Thus, $y = 5 + (y_0 - 5)e^{-2t}$.

Note that in all cases the long-term behavior of all solutions is to approach the equilibrium solution, which is 5 in parts (a) and (c), and 5/2 in part (b). Also, because e^{-2t} decreases faster than e^{-t}, the solutions in (b) and (c) approach equilibrium more quickly than that in (a).

2. Compare the results of this exercise with Exercise 1.

(a) Since $\frac{dy}{dt} = y - 5$, we have

$$\begin{aligned}
\frac{dy/dt}{y-5} &= 1 \\
\frac{d}{dt}[\ln|y-5|] &= 1 \\
\ln|y-5| &= t + C \\
\ln|y-5| &= t + C \\
|y-5| &= e^{t+C} \\
y - 5 &= \pm Ce^{t} \\
y = 5 \pm Ce^{t}.
\end{aligned}$$

Since $y(0) = 5 \pm C = y_0$, we must have $C = \pm(y_0 - 5)$. Thus, $y = 5 + (y_0 - 5)e^{t}$.

(b) Here we have $\frac{dy}{dt} = 2y - 5$, so

$$\begin{aligned}
\frac{dy/dt}{2y-5} &= 1 \\
\frac{1}{2}\frac{d}{dt}[\ln|2y-5|] &= 1 \\
\frac{d}{dt}[\ln|2y-5|] &= 2 \\
|2y-5| &= Ce^{2t} \\
2y - 5 &= \pm Ce^{2t} \\
y &= \frac{5}{2} \pm Ce^{2t}.
\end{aligned}$$

With $y(0) = y_0$, we have $\frac{5}{2} \pm C = y_0$, so $C = \pm(y_0 - 5/2)$. Thus, $y = \frac{5}{2} \pm \left(y_0 - \frac{5}{2}\right) e^{2t}$.

(c) $\frac{dy}{dt} = 2y - 10$, so

$$\begin{aligned}
\frac{dy/dt}{2y-10} &= 1 \\
\frac{1}{2}\frac{d}{dt}[\ln|2y-10|] &= 1 \\
\frac{d}{dt}[\ln|2y-10|] &= 2 \\
|2y-10| &= Ce^{2t} \\
2y - 10 &= \pm Ce^{2t} \\
y &= 5 \pm Ce^{2t}.
\end{aligned}$$

With $y(0) = y_0$, we have $5 \pm C = y_0$, so $C = \pm(y_0 - 5)$. Thus, $y = 5 + (y_0 - 5)e^{2t}$.

Note that in all cases the long-term behavior of all solutions except the equilibrium solution is to diverge from the equilibrium solution, which is 5 in parts (a) and (c), and 5/2 in part (b). Also, because e^{2t} increases faster than e^{t}, the solutions in (b) and (c) diverge from equilibrium more quickly than that in (a).

6. (a) From Example 1, we have $p = 900 - 50e^{t/2}$. Extinction occurs when $p = 0$, so we solve:

$$\begin{aligned} 900 &= 50e^{t/2} \\ 18 &= e^{t/2} \\ t/2 &= \ln 18 \\ t &= 2\ln 18 \\ t &\approx 5.78. \end{aligned}$$

The population will be extinct in about 5.78 months.

(b) We now have $p(0) = p_0 = 900 + c$, so $c = p_0 - 900$. Thus $p = 900 + (p_0 - 900)e^{t/2}$. Setting this equal to zero and solving yields

$$\begin{aligned} (900 - p_0)e^{t/2} &= 900 \\ e^{t/2} &= \frac{900}{900 - p_0} \\ \frac{t}{2} &= \ln\left(\frac{900}{900 - p_0}\right) \\ t &= 2\ln\left(\frac{900}{900 - p_0}\right). \end{aligned}$$

(c) For the population to be extinct in 1 year (12 months), we need

$$\begin{aligned} 0 &= 900 + (p_0 - 900)e^{12/2} \\ p_0 &= 900(e^6 - 1)e^{-6} \\ p_0 &\approx 898. \end{aligned}$$

If the population begins with 898 mice, it will be extinct in about one year.

11. First, let's solve the differential equation $Q' = -rQ$. We have $\frac{Q'}{Q} = -r$, so $\ln|Q| = -rt + C$. Since Q must be positive, we may drop the absolute value bars. Now $Q = Ce^{-rt}$. We don't know what C is, but we do know that $Q(0) = C$ and therefore $Q(\tau) = \frac{C}{2}$. (This is what it means to say that τ is the half-life.) Thus $Ce^{-r\tau} = \frac{C}{2}$; notice that the C's cancel out – the initial amount is irrelevant!

We now have $e^{-r\tau} = \frac{1}{2}$, so $-r\tau - \ln 2$, and $r\tau = \ln 2$, as desired.

12. If the amount is reduced *by* one-quarter, it is reduced *to* three-quarters. The form of a solution to the equation $Q' = -rQ$ is given in Exercise 11, above, as $Q = Ce^{-rt}$. We are told that the half-life is 1620 years, and we know that $1620r = \ln 2$, so $r = \frac{\ln 2}{1620}$. Thus, we need to solve:

$$\begin{aligned} Ce^{-t(\ln 2)/1620} &= \frac{3}{4}C \\ -\frac{t\ln 2}{1620} &= \ln\frac{3}{4} \\ t &= \frac{\ln(4/3)}{\ln 2}\cdot 1620 \\ t &\approx 672.36. \end{aligned}$$

The amount of radium-226 is reduced by 1/4 in about 672 years.

13. (a) We solve as follows:

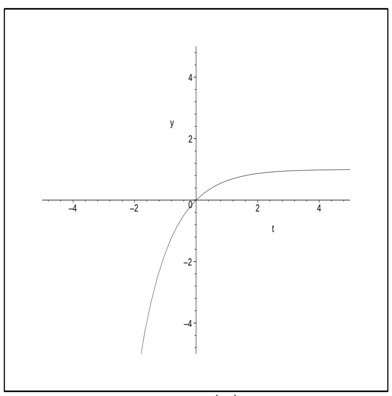
$$
\begin{aligned}
RQ' + \frac{Q}{C} &= V \\
RQ' &= V - \frac{Q}{C} \\
\frac{RQ'}{V - Q/C} &= 1 \\
-CR\frac{d}{dt}\ln|V - Q/C| &= 1 \\
\frac{d}{dt}\ln|V - Q/C| &= -\frac{1}{CR} \\
\ln|V - Q/C| &= -\frac{1}{CR}t + c \\
V - Q/C &= \pm ce^{-t/(CR)} \\
Q &= CV \pm ce^{-t/(CR)}.
\end{aligned}
$$

Whew!

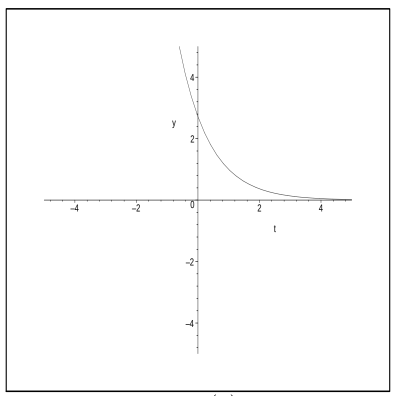
Since $Q(0) = 0$, we have $CV \pm c = 0$, so $c = \mp CV$. Therefore $Q(t) = CV - CVe^{-t/(CR)}$. The graph is below; I set $C = V = R = 1$ so MAPLE could plot it.

(b) Since $\lim_{t\to\infty} Q(t) = CV$, $Q_L = CV$.

(c) Now we are giving an "initial" value of $Q_L = CV$ at $t = t_1$. That is, we have $Q(t) = \pm ce^{-t/(CR)}$ for $t \geq t_1$. At $t = t_1$, this is $CV = \pm ce^{-t_1/(CR)}$, so $c = \pm CVe^{t_1/(CR)}$. This gives $Q(t) = CVe^{t_1/(CR)}e^{-t/(CR)} = CVe^{-(t-t_1)/(CR)}$. I set $t_1 = 1$ for MAPLE's sake.



Part (a)



Part(c)

15. In part (c), the units are grams, so we will use grams throughout.

(a) We are told that $q(0) = 5000$ g, and the pool contains 60000 gallons of water. The concentration of dye at time t is $\frac{q(t)}{60000}$ g/gal, and the dye is removed from 200 gallons per minute. Thus at time t, $\frac{q}{60000}(200) = \frac{q}{300}$ g of dye are being removed per minute. That is, $q'(t) = -\frac{q(t)}{300}$, and $q(0) = 5000$.

(b) We have

$$
\begin{aligned}
\frac{q'(t)}{q(t)} &= -\frac{1}{300} \\
\frac{d}{dt}\ln|q(t)| &= -\frac{1}{300} \\
\ln|q(t)| &= -\frac{t}{300} + C \\
q(t) &= Ce^{-t/300}.
\end{aligned}
$$

Notice that we have dropped the absolute value bars since q must be positive. With $q(0) = C = 5000$, we have $q(t) = 5000e^{-t/300}$.

(c) Since 4 hours is 240 minutes, the amount of dye will be $5000e^{-240/300} \approx 2246.64$ g, making the concentration $2246.64/60000 \approx 0.037$ g/gal. Prepare for some green guests.

(d) We need to have $0.02 = \dfrac{5000e^{-t/300}}{60000} = \dfrac{e^{-t/300}}{12}$. Solving for t gives about 428 minutes, or a little over 7 hours. Bummer!

(e) Let a represent the flow rate. Then $q(t) = 5000e^{-at/60000}$. (See part (a) for the effect of the flow rate; the 300 in the denominator of the exponent came from reducing 200/60000.) With $t = 240$, we need $\dfrac{e^{-240a/60000}}{12} = 0.02$. Solving gives $a \approx 356.78$ gal/min. (The back of the book has 256.78; I believe this is a typo.)