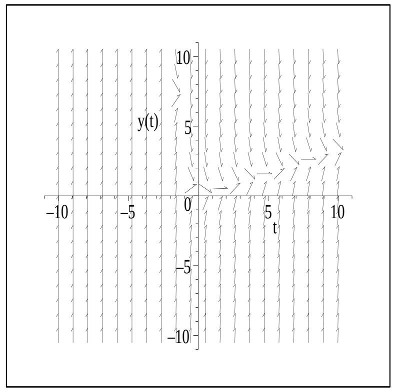
Solutions to Homework Assignment 4

MATH 256-01
Section 2.1, Page 38
Problems: 1, 8, 13-20, 25, 26, 28, 32, 34

1. (a)



(b) It appears that in all cases, the solutions approach ∞ as $t \to \infty$.

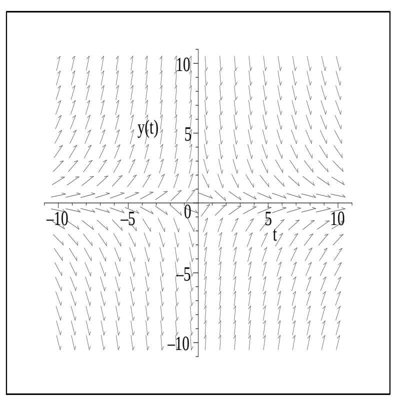
(c) We need an integrating factor $\mu(t) = e^{\int 3\mathrm{d}t} = e^{3t}$. Multiplying both sides by this gives $e^{3t}y' + 3e^{3t}y = te^{3t} + e^t$.

We can see that the left-hand side is $\frac{d}{dt}(e^{3t}y)$. Integrating the right-hand side by parts yields $\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + C$.

Therefore, $e^{3t}y = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + C$ and $y = \frac{1}{3}t - \frac{1}{9} + e^{-2t} + Ce^{-3t}$.

Notice that the last two terms drop out as t becomes large, so the solutions are asymptotic to $\frac{1}{3}t - \frac{1}{9}$.

8. (a)



(b) It appears that solutions converge to 0 as $t \to \infty$.

(c) $y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$. We have an integrating factor of $\mu(t) = e^{\int \frac{4t}{1+t^2}\mathrm{d}t} = e^{2\ln(1+t^2)} = (1+t^2)^2$. This gives

$$
\begin{aligned}
(1+t^2)^2 y' + 4t(1+t^2) &= (1+t^2)^{-1} \\
\frac{d}{dt}[(1+t^2)^2 y] &= \frac{1}{1+t^2} \\
(1+t^2)^2 y &= \arctan(t) + C \\
y &= \frac{\arctan(t) + C^{2}}{1+t^2}.
\end{aligned}
$$

Since $\arctan(t)$ is bounded, this does indeed approach 0 as $t \to \infty$.

13. $\mu(t) = e^{\int -1\mathrm{d}t} = e^{-t}$, so we get $e^{-t}y' - e^{-t}y = 2te^t$. The left-hand side is the derivative of $e^{-t}y$. To integrate the right-hand side, use integration by parts: Let $u = t, dv = e^t$. After integrating, we arrive at $e^{-t}y = 2te^t - 2e^t + C$, so $y = 2te^{2t} - 2e^{2t} + Ce^t$. With $t = 0, y = 1$, and we have $1 = -2 + C$. Thus, $C = 3$ and $y = 2te^{2t} = 2e^{2t} + 3e^t$.

14. $\mu(t) = e^{\int 2\mathrm{d}t} = e^{2t}$, so we get $e^{2t} + 2e^{2t}y = t$, or $\frac{d}{dt}[e^{2t}y] = t$. Thus, $e^{2t}y = \frac{1}{2}t^2 + C$, and $y = \left(\frac{1}{2}t^2 + C\right)e^{-2t}$. With $y(1) = 0$, we get $0 = \left(\frac{1}{2} + C\right)e^{-2}$, so $C = -\frac{1}{2}$. Therefore, $y(t) = \frac{1}{2}e^{-2t}(t^2 - 1)$.

15. Rewrite: $y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$. Now $\mu(t) = e^{\int \frac{2}{t}\mathrm{d}t} = e^{2\ln t} = t^2$. Thus, $t^2y' + 2ty = t^3 - t^2 + t$, so $t^2y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C$, and $y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + Ct^{-2}$. $y(1) = \frac{1}{2}$ implies $C = \frac{1}{12}$, so $y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12}t^{-2}$.

16. $\mu(t) = t^2$ (See number 15.) We have $t^2y' + 2ty = \cos t$, so $t^2y = \sin t + C$ and $y = \frac{\sin t + C}{t^2}$. $y(\pi) = 0$ implies $C = 0$, so $y(t) = \frac{\sin t}{t^2}$.

17. $\mu(t) = e^{\int -2\mathrm{d}t} = e^{-2t}$. $e^{-2t} - 2e^{-2t}y = 1$, so $e^{-2t}y = t + C$ and $y = te^{2t} + Ce^{2t}$. $y(0) = 2 = C$, so $y(t) = te^{2t} + 2e^{2t}$.

18. $\mu(t) = t^2$ (after rewriting; see number 15), so we have $t^2y' + 2ty = t\sin t$. Integrate the right-hand side by parts with $u = t$ and $dv = \sin t$. The result is $t^2y = -t\cos t + \sin t + C$, or $y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$. $y(\pi/2) = 1$ implies $1 = \frac{4}{\pi^2} + \frac{4C}{\pi^2}$, so $C = \frac{\pi^2}{4} - 1$. Thus $y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{\pi^2}{4t^2} - \frac{1}{t^2}$.

19. For this one, it's not hard to see that $\mu(t) = t$. Thus $t^4y' + 4t^3y = te^{-t}$, and integration by parts $(u = t, dv = e^{-t}$ gives $t^4y = -te^{-t} - e^{-t} + C$. Therefore $y = -e^{-t}(t^{-3} + t^{-4}) + Ct^{-4}$. $y(-1) = 0$ implies that $-1 + 1 + C = 0$, so $C = 0$. Therefore $y(t) = -e^{-t}(t^{-3} + t^{-4})$.

20. We have $y' + \left(1 + \frac{1}{t}\right)y = 1$. $\mu(t) = e^{\int 1 + \frac{1}{t}\mathrm{d}t} = e^{t + \ln t} = te^t$. This gives $te^ty' + (t+1)e^ty = te^t$, or $\frac{d}{dt}(te^ty) = te^t$. Thus $te^ty = te^t - e^t + C$, and $y = 1 - \frac{1}{t} + \frac{C}{te^t}$. $y(\ln 2) = 1$ implies that $1 = 1 - \frac{1}{\ln 2} + \frac{C}{2\ln 2}$; thus, $C = 2$. Therefore, $y(t) = 1 - \frac{1}{t} + \frac{2}{te^t}$.

25. We solve. $\mu(t) = e^{\int \frac{1}{2}\mathrm{d}t} = e^{\frac{1}{2}t}$. Thus $e^{\frac{1}{2}t}y' + \frac{1}{2}e^{\frac{1}{2}t}y = 2e^{\frac{1}{2}t}\cos t$. The left-hand side is $\frac{d}{dt}(e^{\frac{1}{2}t}y)$. For the right-hand side, refer to your calculus text under integration by parts, and find an example similar to $\int e^t \cos t dt$. We get $e^{\frac{1}{2}t}y = \frac{8e^{\frac{1}{2}t}}{5}\left(\frac{1}{2}\cos t + \sin t\right) + C$, so $y = \frac{4}{5}(\cos t + 2\sin t) + Ce^{-\frac{1}{2}t}$. $y(0) = -1$ gives $-1 = \frac{4}{5} + C$, so $C = -9/5$, and $y(t) = \frac{4}{5}(\cos t + 2\sin t) - \frac{9}{5}e^{-\frac{1}{2}t}$.

For the maxima, we need the derivative, which we already know is $2\cos t - \frac{1}{2}y$, or

$$2\cos t - \frac{1}{2}\left(\frac{4}{5}(\cos t + 2\sin t) - \frac{9}{5}e^{-\frac{1}{2}t}\right).$$

This is a mess, so I asked MAPLE to set this equal to zero and solve for me. It gave $t \approx 1.36$ and $y \approx 0.82$.

The question remains; is this really a maximum? I will use the second derivative test: $y'' + \frac{1}{2}y' = -2\sin t$, so $y'' = -2\sin t - \frac{1}{2}y' = -2\sin t - \frac{1}{2}\left(2\cos t - \frac{y}{2}\right)$. Evaluating the right-hand side at $t \approx 1.36$ and $y \approx 0.82$, we get $y'' \approx -1.96$; thus, the graph is concave down at this point and we do have a maximum. Whew!

26. We have $\mu(t) = e^{\int \frac{2}{3}\mathrm{d}t} = e^{\frac{2}{3}t}$. This gives $e^{\frac{2}{3}t} + \frac{2}{3}e^{\frac{2}{3}t}y = \left(1 - \frac{1}{2}t\right)e^{\frac{2}{3}t}$, so $\frac{d}{dt}(e^{\frac{2}{3}t}y) = \left(1 - \frac{1}{2}t\right)e^{\frac{2}{3}t}$. Integrating gives

$$e^{\frac{2}{3}t}y = \frac{3}{2}e^{\frac{2}{3}t} - \frac{1}{2}\left(\frac{3}{2}t - \frac{9}{4}\right)e^{\frac{2}{3}t} + C.$$

Therefore, $y = \frac{21}{8} - \frac{3}{4}t + Ce^{-2t/3}$. $y(0) = \frac{21}{8} + C = y_0$, so $C = y_0 - \frac{21}{8}$. This gives us

$$y(t) = \frac{21}{8} - \frac{3}{4}t + \left(y_0 - \frac{21}{8}\right)e^{-2t/3}.$$

For $y = 0$ (where the graph meets the t-axis), we also want $y' = 0$ so we touch but don't cross the t-axis. From the original DE, this must occur at $t = 2$, so we need $y(2) = 0$.

$$\begin{aligned} 0 &= \frac{21}{8} - \frac{3}{4}(2) + \left(y_0 - \frac{21}{8}\right)e^{-2(2)/3} \\ &= \frac{21}{8} - \frac{3}{2} + y_0e^{-4/3} - \frac{21}{8}e^{-4}3 \\ -0.433 &= y_0e^{-4/3} \\ y_0 &= -1.643. \end{aligned}$$

28. From experience, we can see that the integrating factor is e^{-t}. This gives $\frac{d}{dt}(e^{-t}y) = e^{-t} + 3e^{-t}\sin t$. Integrating both sides with respect to t yields $e^{-t}y = -e^{-t} - \frac{3e^{-t}}{2}(\sin t + \cos t) + C$, and $y = -1 - \frac{3}{2}(\sin t + \cos t) + Ce^{-t}$. Since $y(0) = y_0 = -\frac{5}{2} + C$, we have $C = y_0 + \frac{5}{2}$. Thus

$$y = -1 - \frac{3}{2}(\sin t + \cos t) + \left(y_0 + \frac{5}{2}\right)e^{t}.$$

In order for this to remain finite, we need the exponential part to disappear, and the only way to make that happen is to have $y_0 = -\frac{5}{2}$.

32. Let us start with the solutions and work backwards: Let $y(t) = 3 - t + Ce^{-t}$. These are certainly all asymptotic to $3 - t$. We have $y' = -1 - Ce^{-t} = 2 - y - t$, so $y' + y = 2 - t$. I will let you solve this to make sure it works.

34. Here again, we can let $y(t) = 4 - t^2 + Ce^{-t}$. Then $y' = -2t - Ce^{-t} = -y - t^2 + 4 - 2t$, so $y' + y = 4 - 2t - t^2$. Again, you can solve this to confirm it is correct.