Solutions to Homework Assignment 5

MATH 256-01
Section 2.2, Page 45
Problems: 1-9, 11, 12, 21, 23, 30, 31

1.

$$
\begin{aligned}
yy' &= x^2 \\
\frac{1}{2}y^2 &= \frac{1}{3}x^3 + C \\
y &= \pm\sqrt{\frac{2}{3}x^3 + C},
\end{aligned}
$$

for $y \neq 0$.

2.

$$
\begin{aligned}
yy' &= \frac{x^2}{1+x^3} \\
\frac{1}{2}y^2 &= \frac{1}{3}\ln|1+x^3| + C \\
y &= \sqrt{\frac{2}{3}\ln|1+x^3| + C},
\end{aligned}
$$

for $x \neq -1, y \neq 0$.

3.

$$
\begin{aligned}
y' &= -y^2 \sin x \\
\frac{y'}{y^2} &= -\sin x \\
-\frac{1}{y} &= \cos x + C,
\end{aligned}
$$

for $y \neq 0$. We also have the constant solution $y = 0$.

4.

$$
\begin{aligned}
y'(3+2y) &= 3x^2 - 1 \\
3y + y^2 &= x^3 - x + C,
\end{aligned}
$$

$y \neq -\frac{3}{2}$.

5.

$$
\begin{aligned}
y' \sec^2(2y) &= \cos^2 x \\
\frac{1}{2}\tan y &= \int \frac{1}{2} + \frac{\cos(2x)}{2} \mathrm{d}x + C \\
\frac{1}{2}\tan y &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C
\end{aligned}
$$

for all y such that $\cos(2y) \neq 0$. If y is equal to a constant for which $\cos(2y) = 0$, we also get a solution. These constants are $y = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \ldots$.

6.

$$\frac{y'}{\sqrt{1-y^2}} = \frac{1}{x}$$
$$\arcsin y = \ln x + C$$
$$y = \sin(\ln|x| + C)$$

for $|y| < 1, x \neq 0$. We also have the constant solutions $y = \pm 1$.

7.

$$(y + e^y)y' = x - e^{-x}$$
$$\frac{1}{2}y^2 + e^y = \frac{1}{2}x^2 + e^{-x} + C$$

for $y + e^y \neq 0$.

8.

$$(1 + y^2)y' = x^2$$
$$y + \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

for all x, y.

9. (a)

$$\frac{y'}{y^2} = 1 - 2x$$
$$-\frac{1}{y} = x - x^2 + C$$
$$y = \frac{1}{x^2 - x + C}.$$

With $y(0) = -\frac{1}{6}$, we get $C = -6$, so $y = \frac{1}{x^2 - x - 6}$.

(b) See below.

(c) The interval is $(-2, 3)$. (We chose that one of the three options because we need to have $x = 0$ in our domain.)

11. (a)

$$xe^x dx + ydy = 0$$
$$xe^x - e^x + \frac{1}{2}y^2 = C.$$

With $y(0) = 1$, we get $-1 + \frac{1}{2} = C$, so $C = -\frac{1}{2}$. Thus $y(x) = \sqrt{2e^x - 2xe^x + 1}$. (We must choose the positive square root so we get $y(0) = 1$.)

(b) See below.

(c) The interval is approximately $(-\infty, 1.157)$.

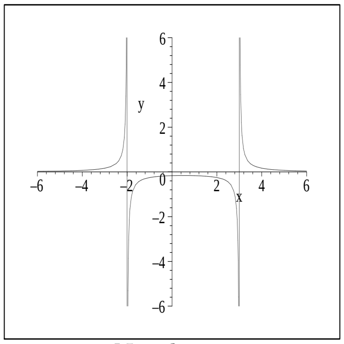
12. (a)

$$\frac{r'}{r^2} = \frac{1}{\theta}$$
$$-\frac{1}{r} = \ln|\theta| + C$$
$$r = -\frac{1}{\ln|\theta| + C}.$$

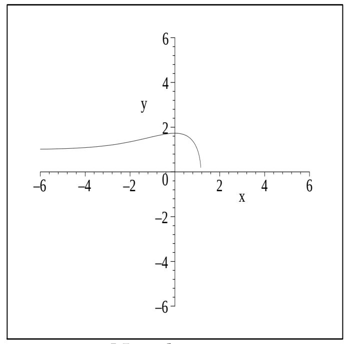
With $r(1) = 2$, we get $2 = -\frac{1}{C}$, so $C = -\frac{1}{2}$ and $r(\theta) = -\frac{1}{\ln|\theta| - \frac{1}{2}} = \frac{2}{1 - 2\ln|\theta|}$.

(b) See below.

(c) The denominator vanishes for $\theta = e^{1/2} = \sqrt{e}$ and $\ln|\theta|$ is undefined for $\theta = 0$, so our interval is $(0, \sqrt{e})$.



Number 9



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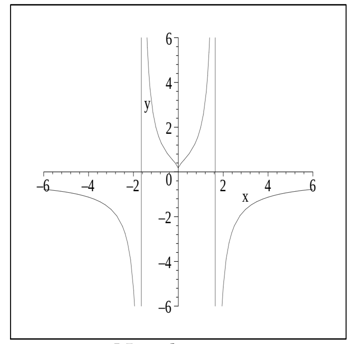
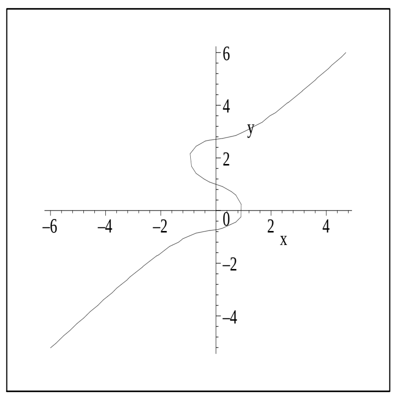


Number 12

21.

$$\begin{aligned} y'(3y^2 - 6y) &= 1 + 3x^2 \\ y^3 - 3y^2 &= x + x^3 + C. \end{aligned}$$

With $y(0) = 1$, we have $1 - 3 = C$, so $C = -2$. Thus $y^3 - 3y^2 = x + x^3 - 2$. The graph appears below.



It looks like the appropriate interval is about $(-1, 1)$.

23.

$$\begin{aligned} \frac{y'}{y^2} &= 2 + x \\ -\frac{1}{y} &= 2x + \frac{1}{2}x^2 + C \\ y &= -\frac{2}{x^2 + 4x + C}. \end{aligned}$$

With $y(0) = 1$, we get $1 = -\dfrac{2}{C}$, so $C = -2$. Thus $y(x) = -\dfrac{2}{x^2 + 4x - 2}$. We get $y' = 0$ when $2y^2 + xy^2 = 0$, or when $y = 0$ or $x = -2$. Since $y \neq 0$ for any x, our only critical number is $x = -2$. The first derivative test gives $y' < 0$ if $x < -2$ and $y' > 0$ if $x > -2$, so $x = -2$ gives a minimum.

30. (a) Divide numerator and denominator by x.

(b) Since $y = xv$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

(c) Now $v + x\dfrac{dv}{dx} = \dfrac{dy}{dx} = \dfrac{(y/x) - 4}{1 - (y/x)} = \dfrac{v - 4}{1 - v}$.Subtracting v from both sides gives $x\dfrac{dv}{dx} = \dfrac{v^2 - 4}{1 - v}$.

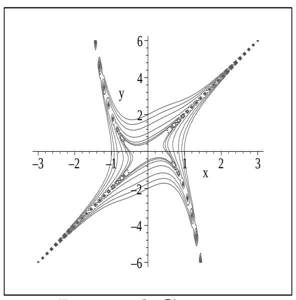
(d)

$$\begin{aligned}
\frac{1-v}{v^2-4}v' &= \frac{1}{x} \\
\left(\frac{-1}{4(v-2)} + \frac{-3}{4(v+2)}\right)v' &= \frac{1}{x} \\
-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| &= \ln|x| + C \\
|v-2|^{-1/4}|v+2|^{-3/4} &= C|x| \\
\frac{1}{|v-2||v+2|^3} &= Cx^4.
\end{aligned}$$

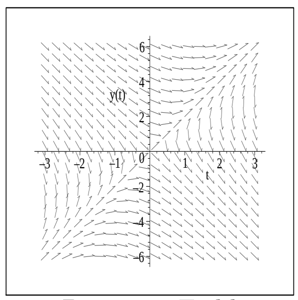
(e)

$$\begin{aligned}
\frac{1}{|(y/x)-2||(y/x)+2|^3} &= Cx^4 \\
\frac{1}{|y-2x||y+2x^3|} &= C \\
|y-2x||y+2x^3| &= C.
\end{aligned}$$

(f)



Integral Curves



Direction Field

The symmetry does show itself.

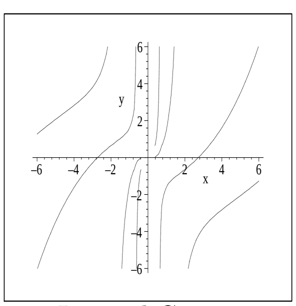
31. (a) $y' = 1 + (y/x) + (y/x)^2$, so this equation is homogeneous.

(b) Let $v = y/x$, so again $y = vx$ and $y' = xv' + v$. Thus

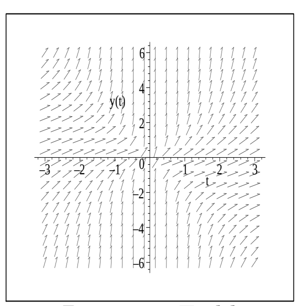
$$\begin{aligned}
xv' + v &= 1 + v + v^2 \\
xv' &= 1 + v^2 \\
\frac{v'}{1+v^2} &= \frac{1}{x} \\
\arctan v &= \ln|x| + C.
\end{aligned}$$

Substituting for v, we get $\arctan(y/x) = \ln|x| + C$.

(c)



Integral Curves



Direction Field

We do seem to have symmetry about the origin.