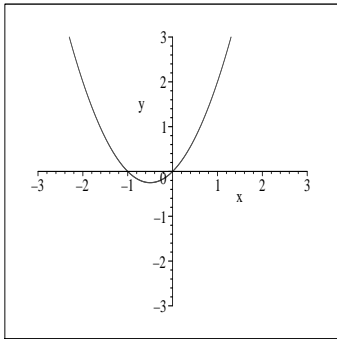


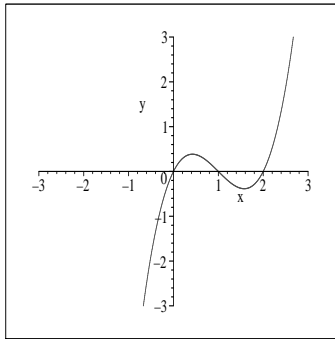
Solutions to Homework Assignment 7

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Section 2.5, Page 84
Problems: 1-9, 15, 26

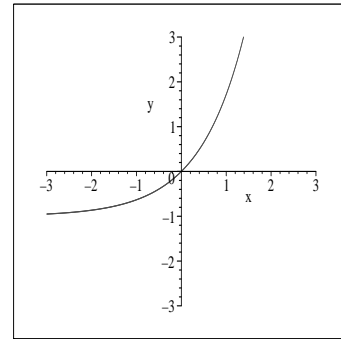
1. The graph is below. The equilibrium points are at $y = 0$ and $y = -\frac{a}{b}$, but since $y_0 \geq 0$, only $y = 0$ is a solution. Notice that $y' > 0$ for $y > 0$ and $y' < 0$ for $-\frac{b}{a} < y < 0$. This means that if y is positive, it grows (and therefore gets farther away from 0); if y is negative, it gets smaller (and therefore gets farther away from 0). Thus, 0 is an unstable equilibrium point.
2. The graph is the same as in Exercise 1. This time, both $y = 0$ and $y = -\frac{a}{b}$ are solutions. The solution $y = 0$ is still unstable (for the same reasons as above), but $y = -\frac{a}{b}$ is a stable solution: if $y < -\frac{a}{b}$, y grows to reach $-\frac{a}{b}$; if $-\frac{a}{b} < y < 0$, then y decreases down to $-\frac{a}{b}$.
3. The graph is below. The equilibrium points are 0, 1, and 2. We can see that 0 and 2 are unstable; if $y < 0$ or $1 < y < 2$, then y decreases away from 0 or 2, and if $1 > y > 0$ or $y > 2$, y increases away from 0 or 2. On the other hand, 1 is a stable solution since for $0 < y < 1$, y increases up to 1, and for $1 < y < 2$, y decreases down to 1.
4. The graph is below. The only equilibrium solution is $y = 0$, and it is unstable since for $y < 0$, y decreases away from 0, and for $y > 0$, y increases away from 0.
5. Again, the only equilibrium solution is $y = 0$, but this time it is stable since for $y < 0$, y increases up to 0, and for $y > 0$, y decreases down to 0.
6. The only equilibrium solution is $y = 0$, and it is stable since for $y < 0$, y increases up to 0, and for $y > 0$, y decreases down to 0.



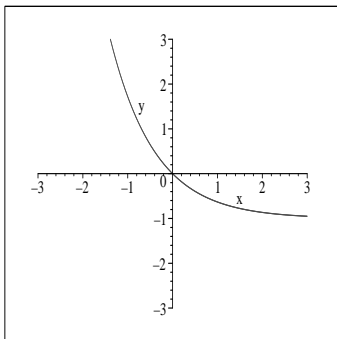
Numbers 1 and 2



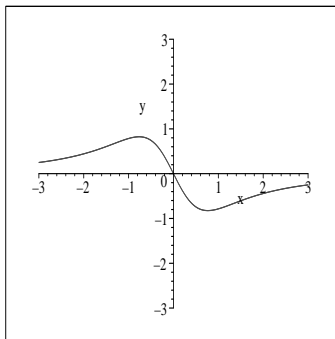
Number 3



Number 4



Number 5



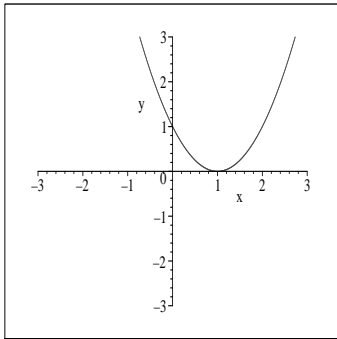
Number 6

7. (a) It is clear that $y = 1$ is the only stable point since $y = 1$ is the only solution of $k(1 - y)^2 = 0$.
 (b) Since $y' > 0$ for $y \neq 1$, y is increasing in t for all $y \neq 1$.
 (c) This is separable:

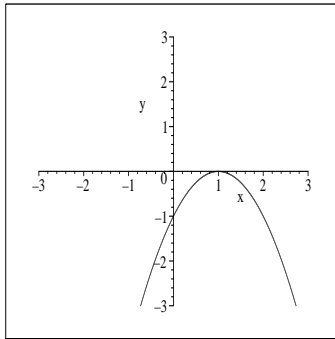
$$\begin{aligned}\frac{y'}{k(1-y)^2} &= 1 \\ \frac{1}{k(1-y)} &= t + C \\ k(1-y) &= \frac{1}{t+C} \\ y &= 1 - \frac{1}{k(t+C)}.\end{aligned}$$

With $y(0) = y_0$, we have $y_0 = 1 - \frac{1}{kC}$, so $C = \frac{1}{k(1-y_0)}$. Therefore, our solution is $y = 1 - \frac{1}{k(t + 1/(k(1-y_0)))}$, or $y = 1 - \frac{1-y_0}{k(1-y_0)t + 1}$. This shows that if $y_0 = 1$, then $y = 1$ for all t . Also, $y' = \frac{k(1-y_0)^2}{(k(1-y_0)t + 1)^2}$, so $y' > 0$ as long as the denominator is nonzero.

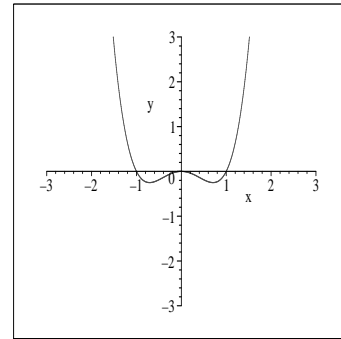
8. The only equilibrium point is $y = 1$. Since $y' < 0$ for $y \neq 1$, $y = 1$ must be semistable: solutions below $y = 1$ decrease away from 1, and solutions above 1 decrease down to 1.
 9. This one has three equilibrium solutions: $y = -1, y = 0$, and $y = 1$. We can see from the graph that $y = -1$ is stable, $y = 0$ is semistable, and $y = 1$ is unstable.



Number 7



Number 8



Number 9

15. From the text and class, we have $y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$.

- (a) With $y_0 = \frac{K}{3}$, this becomes $y(t) = \frac{K^2/3}{K/3 + (2K/3)e^{-rt}} = \frac{K}{1 + 2e^{-rt}}$. We want to set this equal to $2K/3$ and solve for t .

$$\begin{aligned}\frac{K}{1 + 2e^{-rt}} &= \frac{2K}{3} \\ 3 &= 2 + 4e^{-rt} \\ \frac{1}{4} &= e^{-rt} \\ t &= \frac{\ln 4}{r}.\end{aligned}$$

With $r = 0.025$, this becomes $t \approx 55.45$ years.

(b) With $y_0 = K\alpha$, we get $y(t) = \frac{K\alpha}{\alpha + (1-\alpha)e^{-rt}}$. We want to set this equal to $K\beta$ and solve for t .

$$\begin{aligned}\frac{K\alpha}{\alpha + (1-\alpha)e^{-rt}} &= K\beta \\ \alpha &= \beta(\alpha + (1-\alpha)e^{-rt}) \\ \frac{\alpha}{\beta} - \alpha &= (1-\alpha)e^{-rt} \\ \frac{\alpha(1-\beta)}{\beta(1-\alpha)} &= e^{-rt} \\ t &= -\frac{1}{r} \ln\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) \\ t &= \frac{1}{r} \ln\left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right).\end{aligned}$$

Whew! With $r = 0.025$, $\alpha = 0.1$, and $\beta = 0.9$, this becomes 175.78 years.

26. Without loss of generality, we may assume $p < q$.

(a) We have two equilibrium solutions: $x = p$ and $x = q$. The graph of x' vs. x is an upward-opening parabola with positive x -intercepts p and q . With $x(0) = 0$, we increase up to p , which is a stable equilibrium. The equation is separable; we get

$$\begin{aligned}\frac{x'}{(p-x)(q-x)} &= \alpha \\ \frac{1/(q-p)}{p-x} + \frac{1/(p-q)}{q-x} &= \alpha \\ \frac{-1}{q-p} \ln(p-x) + \frac{-1}{p-q} \ln(q-x) &= \alpha t + C \\ \frac{1}{q-p} \ln\left(\frac{q-x}{p-x}\right) &= \alpha t + C \\ \frac{q-x}{p-x} &= Ce^{(q-p)\alpha t}.\end{aligned}$$

Since $x(0) = 0$, we have $\frac{q}{p} = C$.

Now

$$\begin{aligned}q-x &= (p-x)Ce^{(q-p)\alpha t} \\ x &= \frac{pCe^{(q-p)\alpha t} - q}{Ce^{(q-p)\alpha t} - 1} \\ x &= \frac{pq(e^{(q-p)\alpha t} - 1)}{qe^{(q-p)\alpha t} - p}.\end{aligned}$$

Notice that since $q > p$, this does approach p as $t \rightarrow \infty$. If $q < p$, this approaches q .

(b) The solution $x = p$ is semistable, and values below p approach p as $t \rightarrow \infty$. This is also separable:

$$\begin{aligned}\frac{x'}{(p-x)^2} &= \alpha \\ f1p-x &= \alpha t + C \\ x &= p - \frac{1}{\alpha t + C}.\end{aligned}$$

Since $x(0) = 0$, $C = \frac{1}{p}$. Therefore, $x(t) = p - \frac{p}{\alpha pt + 1}$. This does approach p as $t \rightarrow \infty$.