Solutions to Homework Assignment 8

MATH 256-01
Section 2.6, Page 95
Problems: 1-14, 18-22, 25-30

1. $M = 2x + 3, M_y = 0; N = 2y - 2, N_x = 0$. Since $M_y = N_x$, this equation is exact. We have

$$\begin{aligned}\psi_x &= M\\ \psi_x &= 2x + 3\\ \psi &= x^2 + 3x + h(y).\end{aligned}$$

This gives $N = \psi_y = h'(y) = 2y - 2$, so $\psi = y^2 - 2y + g(x)$. In order for the expressions for ψ to be equal, we need $g(x) = x^2 + 3x$, so $x^2 + 3x + y^2 - 2y = C$ defines y implicitly as a function of x.

2. $M_y = 4, N_x = 2$, so this equation is not exact.

3. $M_y = -2x, N_x = -2x$, so this equation is exact.

$$\begin{aligned}\psi_x &= M\\ \psi_x &= 3x^2 - 2xy + 2\\ \psi &= x^3 - x^2y + 2x + h(y).\end{aligned}$$

Now $\psi_y = -x^2 + h'(y) = -x^2 + 6y^2 + 3$, so $h'(y) = 6y^2 + 3$. Thus $h(y) = 2y^3 + 3y + g(x)$. Therefore, $psi = x^3 - x^2y + 2x + 2y^3 + 3y + g(x)$. In this case, since $\psi_x = 3x^2 - 2xy + 2$, we must have $g(x) = C$, so $x^3 - x^2y + 2x + 2y^3 + 3y + C = k$ defines y implicitly as a function of x that solves our DE. We combine constants to get $x^3 - x^2y + 2x + 2y^3 + 3y = C$.

4. $M_y = 4xy + 2$, $N_x = 4xy + 2$, so this equation is exact. We get $\psi = x^2y^2 + 2xy + h(y)$, so $\psi_y = 2x^2y + 2x + h'(y)$. Therefore, $h'(y) = 0$, and $h(y) = C$. Therefore, $x^2y^2 + 2xy = C$ defines our solution implicitly.

5. This is equivalent to $(ax + by) + (bx + cy)y' = 0$. This has $M_y = b$ and $N_x = b$, so it is exact. We have $\psi_x = ax + by$, so $\psi = \frac{1}{2}ax^2 + bxy + h(y)$. $\psi_y = bx + h'(y)$, so $h'(y) = cy$. Thus $h(y) = \frac{1}{2}cy^2 + C$, and $\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2 + C = k$ defines our solution. This may be written as $ax^2 + 2bxy + cy^2 = C$.

6. This is equivalent to $(ax - by) + (bx - cy)y' = 0$. This has $M_y = -b, N_x = b$, so it is not exact.

7. $M_y = e^x \cos y - 2\sin x, N_x = e^x \cos y - 2\sin x$, so this is exact. We get $\psi = e^x \sin y + 2y\cos x + h(y)$. $\psi_y = e^x \cos y + 2\cos x + h'(y)$, so $h'(y) = 0$ and $h(y) = C$. We have $e^x \sin y + 2y\cos x = C$ as our solution.

8. $M_y = e^x \cos y + 3, N_x = e^x \sin y - 3$, so this is not exact.

9. I asked MAPLE to find M_y and N_x for me; this equation is exact. Integrating M with respect to x gives $\psi = e^{xy}\cos 2x + x^2 + h(y)$ (from MAPLE, again). Thus $\psi_y = xe^{xy}\cos 2x + h'(y)$, so $h'(y) = -3$ and $h(y) = -3y + C$. Our equation is therefore $e^{xy}\cos 2x + x^2 - 3y = C$.

10. $M_y = 1/x, N_x = 1/x$, so this equation is exact. We get $\psi = y\ln x + 3x^2 + h(y)$ (we are told that $x > 0$), so $\psi_y = \ln x + h'(y)$. Thus $h'(y) = -2$, so $h(y) = -2y + C$. We now have $y\ln x + 3x^2 - 2y = C$.

11. $M_y = x/y + x, N_x = y/x + y$, so this is not exact.

12. MAPLE again assures me that this equation is exact. $\psi = \dfrac{-1}{(x^2+y^2)^{1/2}} + h(y)$, so $\psi_y = \dfrac{y}{(x^2+y^2)^{3/2}} + h'(y)$. Thus $h'(y) = 0$ and $h(y) = C$. We arrive at $\dfrac{-1}{(x^2+y^2)^{1/2}} = C$, or $\sqrt{x^2+y^2} = C$ or even $x^2 + y^2 = C$ (an equation of a circle!).

13. This is exact with $\psi = x^2 - xy + h(y)$, so $\psi_y = -x + h'(y)$; thus, $h'(y) = 2y$ and $h(y) = y^2 + C$. This gives $x^2 - xy + y^2 = C$. With $y(1) = 3$, we get $1^2 - 1 \cdot 3 + 3^2 = C$, so $C = 7$. Our equation is therefore $x^2 - xy + y^2 = 7$. Since this is quadratic in y, we can solve for y explicitly: $y = \dfrac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2}$.

To have $y(1) = 3$ we must choose the positive root, so $y(x) = \dfrac{x + sqrt 28 - 3x^2}{2}$. This is valid provided $28 - 3x^2 \geq 0$, or $|x| < \sqrt{28/3}$.

14. This is exact with $\psi = 3x^3 + xy - x + h(y)$, so $\psi_y = x + h'(y)$. Thus, $h'(y) = -4y$ and $h(y) = -2y^2 + C$. We get $3x^3 + xy - x - 2y^2 = C$. With $y(1) = 0$, this becomes $3 - 1 = C$, so $C = 2$. We therefore have $3x^3 + xy - x - 2y^2 = 2$.

We can also solve this explicitly for y: $2y^2 - xy + (2 + x - 3x^3) = 0$, so $y = \dfrac{x \pm \sqrt{x^2 - 4(2)(2 + x - 3x^3)}}{4} = \dfrac{x - \sqrt{24x^3 + x^2 - 8x - 16}}{4}$. (We need the negative root.) The radicand is positive (and hence the solution makes sense) for $x > 0.9523$-ish, according to my TI-86.

18. $M_y = 0$ and $N_x = 0$, so separable equations are exact.

19. $M_y = 3x^2y^2, N_x = 1 + y^2$, so this is not exact. Multiplying through by μ gives $x + \dfrac{1 + y^2}{y^3} y' = 0$. Since this equation is separable, it is exact. We get $(y^{-3} + y^{-1})y' = -x$, so $-\dfrac{1}{2}y^{-2} + \ln|y| = -\dfrac{1}{2}x^2 + C$. The constant function $y = 0$ is also a solution.

20. MAPLE says $M_y \neq N_x$, so this equation is not exact. Multiplying by μ gives $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)y' = 0$. Now $M_y = e^x \cos y - 2 \sin x$ and $N_x = e^x \cos y - 2 \sin x$, so the new equation is exact. We have

$$\begin{aligned} \psi_x &= e^x \sin y - 2y \sin x \\ \psi &= e^x \sin y + 2y \cos x + h(y). \end{aligned}$$

Now $\psi_y = e^x \cos y + 2 \cos x + h'(y)$, so $h'(y) = 0$ and $h(y) = C$. Our solution is defined implicitly by $e^x \sin y + 2y \cos x = C$.

21. $M_y = 1$ and $N_x = 2$, so this is not exact. Multiplying through by y gives $y^2 + (2xy - y^2 e^y)y' = 0$, with $M_y = 2y$ and $N_x = 2y$, so this one is exact. $\psi = xy^2 + h(y)$, so $\psi_y = 2xy + h'(y)$, and $h'(y) = -y^2 e^y$. Thus $h(y) = -y^2 e^y + 2ye^y - 2e^y + C$ and $xy^2 - y^2 e^y + 2ye^y - 2e^y = C$.

22. $M_y = (x + 2)\cos y, N_x = \cos y$, so this is not exact. Multiplying through by μ gives $x(x + 2)e^x \sin y + (x^2 e^x \cos y)y' = 0$. This is exact, according to MAPLE.

We have $\psi = x^2 e^x \sin y + h(y)$ (according to MAPLE), so $\psi_y = x^2 e^x \cos y + h'(y)$. Thus $h'(y) = 0$. Therefore, our equation is $x^2 e^x \sin y = C$.

For each of 25-30, we compute $\dfrac{M_y - N_x}{-M}$ and $\dfrac{M_y - N_x}{N}$ (if necessary). I used MAPLE for these computations.

Number	$\frac{M_y - N_x}{N}$	$\frac{M_y - N_x}{-M}$
25.	3	Unnecessary
26.	-1	Unnecessary
27.	$\frac{1}{y\sin y - x}$	$\frac{1}{y}$
28.	$\frac{2y-1}{e^{-2y} - 2xy}$	$\frac{2y-1}{y}$
29.	$\frac{-e^x \cos y}{e^x \cos y + 2y}$	$\cot y$
30.	$\frac{-2(4x^3+3y)}{y(3x+4y^3)}$	$\frac{2}{y}$

Now we can use this data to find the appropriate integrating factors.

25. We have $\mu'(x) = 3\mu$, so our integrating factor is $\mu = e^{3x}$. We get $(3x^2y+2xy+y^3)e^{3x}+(x^2+y^2)e^{3x}y' = 0$. This gives $\psi = \frac{1}{3}y(3x^2+y^2)e^{3x}+h(y)$. Now $\psi_y = e^{3x}(x^2+y^2)$, so $h'(y) = 0$ and $h(y) = C$. Our solution is therefore defined implicitly by $\frac{1}{3}y(3x^2+y^2)e^{3x} = C$ or just $y(3x^2+y^2)e^{3x} = C$.

26. We have $\mu'(x) = -\mu$, so $\mu(x) = e^{-x}$. We get $(e^x+e^{-x}y-e^{-x})-e^{-x}y' = 0$. Thus $\psi = e^x - e^{-x}y + e^{-x} + h(y)$ and $\psi_y = -e^{-x} + h'(y)$, so $h'(y) = 0$ and $h(y) = C$. Our solution is therefore defined implicitly by $e^x - e^{-x}y + e^{-x} = C$. This time we can solve for y: $y = e^{2x} + 1 + Ce^x$.

27. We have $\mu'(y) = \frac{1}{y}\mu$, so $\mu = y$. We get $y + (x - y\sin y)y' = 0$. Thus $\psi = xy + h(y)$, so $\psi_y = x + h'(y)$. Therefore, $h'(y) = -y\sin y$ and $h(y) = y\cos y - \sin y + C$. Our solution is defined implicitly by $xy + y\cos y - \sin y = C$.

28. We have $\mu'(y) = \left(2 - \frac{1}{y}\right)\mu$, so $\ln\mu = 2y - \ln y$ and $\mu = \frac{e^{2y}y}{.}$ Multiplying through by μ gives $e^{2y} + (2xe^{2y} - 1/y)y' = 0$. Therefore, $\psi = xe^{2y} + h(y)$ and $\psi_y = 2xe^{2y} + h'(y)$, and $h'(y) = -1/y$. Thus $h(y) = -\ln|y| + C$, giving $xe^{2y} - \ln|y| = C$. The constant function $y = 0$ is also a solution.

29. We have $\mu'(y) = \mu\cot y$, so $\frac{\mu'}{\mu} = \frac{\cos y}{\sin y}$. Thus $\ln\mu = \ln\sin y$, so $\mu = \sin y$. Multiplying through by μ gives $e^x\sin y + (e^x\cos y + 2y)y' = 0$. Thus $\psi = e^x\sin y + h(y)$, so $\psi_y = e^x\cos y + h'(y)$. Therefore, $h'(y) = 2y$, so $h(y) = y^2 + C$. We have our solution implicitly defined by $e^x\sin y + y^2 = C$.

30. $\mu'(y) = \frac{2}{y}\mu(y)$, so $\mu(y) = y^2$. Multiplying through by y gives $[4x^3+3y] + [3x+4y^3]y' = 0$. $\psi = x^4 + 3xy + h(y)$ and $\psi_y = 3x + h'(y)$. Therefore, $h'(y) = 4y^3$ and $h(y) = y^4 + C$. Our solution is defined implicitly by $x^4 + 3xy + y^4 = C$.