Solutions to Homework Assignment 10

MATH 256-01
Section 3.1, Page 136
Problems: 1-17 odd, 20, 21, 29-43 odd

1. The characteristic equation is $r^2 + 2r - 3 = 0$, so $(r+3)(r-1) = 0$. Thus $r = -3$ or $r = 1$, and the general solution is $c_1 e^{-3t} + c_2 e^t$.

3. $6r^2 - r - 1 = 0$ implies $(3r+1)(2r-1) = 0$, so $r = -1/3$ or $r = 1/2$. The general solution is $c_1 e^{-t/3} + c_2 e^{t/2}$.

5. $r^2 + 5r = 0$, so $r = 0$ or $r = -5$. The general solution is $c_1 + c_2 e^{-5t}$.

7. Here we have $r^2 - 9r + 9 = 0$, so $r = \dfrac{9 \pm \sqrt{45}}{2} = \dfrac{9 \pm 3\sqrt{5}}{2}$. The general solution is therefore $c_1 e^{(9+3\sqrt{5})t/2} + c_2 e^{9-3\sqrt{5})t/2}$.

9. $r^2 + r - 2 = 0$, so $r = -2$ or $r = 1$. The general solution is $c_1 e^{-2t} + c_2 e^t$. Since $y(0) = 1$, $c_1 + c_2 = 1$. Since $y'(0) = 1$, $-2c_1 + c_2 = 1$. This gives $c_1 = 0$ and $c_2 = 1$, so $y(t) = e^t$. Thus $y \to \infty$ and $t \to \infty$.

11. $6r^2 - 5r + 1 = 0$, so $r = 1/3$ or $r = 1/2$. The general solution is $c_1 e^{t/3} + c_2 e^{t/2}$. Since $y(0) = 4, c_1 + c_2 = 4$. Since $y'(0) = 0$, $\dfrac{c_1}{3} + \dfrac{c_2}{2} = 0$. This gives $c_2 = -8$ and $c_1 = 12$. Therefore, $y(t) = 12e^{t/3} - 8e^{t/2}$. The larger exponent will dominate, so $y \to \infty$ and $t \to \infty$.

13. $r^2 + 5r + 3 = 0$, so $r = \dfrac{-5 \pm \sqrt{13}}{2}$. The general solution is $c_1 e^{(\sqrt{13}-5)t/2} + c_2 e^{(-\sqrt{13}-5)t/2}$. Since $y(0) = 1$, $c_1 + c_2 = 1$. Since $y'(0) = 0$, $\dfrac{\sqrt{13}-5}{2}c_1 + \dfrac{-\sqrt{13}-5}{2}c_2 = 0$. We have $c_1 = 1 - c_2$, so $\dfrac{\sqrt{13}-5}{2}(1-c_2) + \dfrac{-\sqrt{13}-5}{2}c_2 = 0$. Thus $\sqrt{13}c_2 = \dfrac{\sqrt{13}-5}{2}$, and $c_2 = \dfrac{\sqrt{13}-5}{2\sqrt{13}} = \dfrac{13-5\sqrt{13}}{26}$. This also gives $c_1 = \dfrac{13+5\sqrt{13}}{26}$, so the solution is $\dfrac{13+5\sqrt{13}}{26}e^{(\sqrt{13}-5)t/2} + \dfrac{13-5\sqrt{13}}{26}e^{(-\sqrt{13}-5)t/2}$.

15. $r^2 + 8r - 9 = 0$, so $r = -9$ or $r = 1$. The general solution is $c_1 e^{-9t} + c_2 e^t$. Since $y(1) = 1$, $c_1 e^{-9} + c_2 e = 1$. Since $y'(1) = 0$, $-9c_1 e^{-9} + c_2 e = 0$. Thus $c_2 = 9c_1 e^{-10}$, so $c_1 e^{-9} + 9c_1 e^{-9} = 1$, so $c_1 = \dfrac{1}{10}e^9$. This gives $c_2 = \dfrac{9}{10}e^{-1}$, so the general solution is $\dfrac{1}{10}e^{-9t+9} + \dfrac{9}{10}e^{t-1}$. We have $y \to \infty$ as $t \to \infty$.

17. We just need a characteristic equation with solutions 2 and -3; say, $(r-2)(r+3) = 0$, or $r^2 + r - 6 = 0$. This corresponds to the ODE $y'' + y' - 6y = 0$.

20. We have $2r^2 - 3r + 1 = 0$, so $r = 1$ or $r = 1/2$. The general solution is $c_1 e^t + c_2 e^{t/2}$. Since $y(0) = 2$, $c_1 + c_2 = 2$. Since $y'(0) = 1/2$, $c_1 + c_2/2 = 1/2$. Solving gives $c_2 = 3$ and $c_1 = -1$, so $y(t) = -e^t + 3e^{t/2}$. This is zero when $3e^{t/2} = e^t$, or $e^{t/2} = 3$, which is $t = 2\ln(3)$. For the maximum, we need $y'(t) = -e^t + \dfrac{3}{2}e^{t/2}$. This is zero when $e^{t/2} = 3/2$, or $t = 2\ln(3/2)$. An examination of the graph shows that this gives a maximum; its value is $y(2\ln(3/2)) = -(3/2)^2 + 9/2 = 9/4$.

21. $r^2 - r - 2 = 0$, so $r = -1$ or $r = 2$. The general solution is $c_1 e^{-t} + c_2 e^{2t}$. Since $y(0) = \alpha, c_1 + c_2 = \alpha$. Since $y'(0) = 2$, $-c_1 + 2c_2 = 2$. Thus $c_2 = \dfrac{\alpha+2}{3}$ and $c_1 = \dfrac{2\alpha-2}{3}$. Therefore, $y(t) = \dfrac{2\alpha-2}{3}e^{-t} + \dfrac{\alpha+2}{3}e^{2t}$. In order to prevent $y \to \pm\infty$ as $t \to \infty$, we need to eliminate the exponentially growing term. To do this, let $\alpha = -2$.

29. Put $v = y', v' = y$. Then $tv' + v = 1$. Thus $\dfrac{d}{dt}(tv) = 1$, so $tv = t + C$, and $v = 1 + \dfrac{c_1}{t}$. Since $v = y'$, we get $y = t + c_1 \ln t + c_2$. (We are told $t > 0$.)

31. With $v = y', v' = y''$, we have $2t^2v' + v^3 = 2tv$, so $v(v^2 - 2t) + 2t^2v' = 0$. $M_v = 3v^2 - 2t$ and $N_t = 4t$, so this is not exact. However, $\dfrac{M_y - N_t}{-M} = \dfrac{3v^2 - 6t}{-v(v^2 - 2t)} = -\dfrac{3}{v}$, which is a function of v only. We may therefore use $\mu = v^{-3}$ as an integrating factor: $(1 - 2t/v^2) + (2t^2/v^3)v' = 0$ is exact. Now $\psi = t - t^2/v^2 + h(v)$, so $\psi_v = N = 2t^2/v^3 + h'(v)$, giving $h'(v) = 0$. Thus our solution is $t - \dfrac{t^2}{v^2} = c_1$, so $v = \pm\dfrac{t}{\sqrt{t + c_1}}$. This is $y' = \pm\dfrac{t}{\sqrt{t + c_1}}$.

To integrate, let $u = t + c_1$. We get $\pm\dfrac{u - c_1}{\sqrt{u}} = \pm(u^{1/2} - c_1u^{-1/2})$. Integration gives

$$\begin{aligned}\pm\left(\frac{2}{3}u^{3/2} - 2c_1u^{1/2}\right) + c_2 &= \pm\left(\frac{2}{3}(t + c_1)^{3/2} - 2c_1(t + c_1)^{1/2}\right) + c_2 \\ &= \pm\sqrt{t + c_1}\left(\frac{2}{3}(t + c_1) - 2c_1\right) + c_2 \\ &= \pm\frac{2}{3}\sqrt{t + c_1}(t - 2c_1) + c_2.\end{aligned}$$

Whew! Also, since we divided by $v = y'$, we need to check $y' = 0$. A quick inspection shows that $y = c$ is also a solution.

33. Letting $v = y', v' = y''$ gives $t^2v' = v^2$, so $\dfrac{v'}{v^2} = \dfrac{1}{t^2}$. Note $v = 0$ is a solution, so $y = c$ is a solution. Now $-\dfrac{1}{v} = -\dfrac{1}{t} + c_1$, so $y' = v = \dfrac{t}{1 - c_1t} = \dfrac{-1}{c_1} + \dfrac{1}{c_1(1 - c_1t)}$. Integration gives $y = -\dfrac{t}{c_1} + \dfrac{1}{c_1^2}\ln|1 - c_1t| + c_2$.

35. Let $v = y'$. Then we have $v' = y''$, so $v(dv/dy) = -y$. Therefore, $\dfrac{1}{2}v^2 = -\dfrac{1}{2}y^2 + c_1$, and $v = \sqrt{c_1 - y^2}$. Now $y' = \sqrt{c_1 - y^2}$, so $\dfrac{y'}{\sqrt{c_1}\sqrt{1 - (y/\sqrt{c_1})^2}} = 1$. Integration gives $\arcsin(y/\sqrt{c_1}) = t + c_2$, so $y = \sqrt{c_1}\sin(t + c_2)$.

37. Let $v = y'$. Then $v' = y''$, so $v(dv/dy) = \dfrac{1}{2y^2} - \dfrac{v^2}{y}$. Thus $(2yv^2 - 1) + 2y^2v\dfrac{dv}{dy} = 0$. Since $M_v = 4yv$ and $N_y = 4yv$, this is exact. Therefore, $\psi = y^2v^2 - y + h(v)$ and $\psi_v = N = 2y^2v + h'(v)$, so $h'(v) = 0$. This gives

$$\begin{aligned}y^2v^2 - y &= c_1 \\ y^2v^2 - 1 &= c_1 + y \\ v^2 &= \frac{(c_1 + y)}{y^2} \text{ (new } c_1) \\ v &= \pm\frac{\sqrt{c_1 + y}}{y}.\end{aligned}$$

Since $v = y'$, we have $y' = \pm\dfrac{\sqrt{c_1 + y}}{y}$. Thus $\dfrac{yy'}{\sqrt{y + c_1}} = \pm 1$. Integrating gives $\dfrac{2}{3}(y + c_1)^{3/2} - 2c_1(y + c_1)^{1/2} = \pm t + c_2$. Thus $t + c_2 = \pm\left[(y + c_1)^{1/2}\left(\dfrac{2}{3}(y + c_1) - 2c_1\right)\right] = \pm\dfrac{2}{3}(y + c_1)^{1/2}(y - 2c_1)$.

39. Let $v = y'$, so $y'' = v(dv/dy)$. We have $(v^2 - 2e^{-y}) + v\dfrac{dv}{dy} = 0$. This is not exact, but $\dfrac{M_v - N_y}{N} = \dfrac{2v - 0}{v} = 2$ is a function of y alone, so $\mu = e^{2y}$ is an integrating factor. We get $(e^{2y}v^2 - 2e^y) + ve^{2y}\dfrac{dv}{dy} = 0$, which is exact. Thus $\psi = \dfrac{1}{2}v^2e^{2y} + h(y)$, so $\psi_y = M = v^2e^{2y} + h'(y)$, giving $h'(y) = -2e^y$. Thus $h(y) = -2e^y$, and our equation is $\dfrac{1}{2}v^2e^{2y} - 2e^y = c_1$.

Now must solve for $v : v = \pm\sqrt{c_1 e^{-2y} + 4e^{-y}} = \dfrac{\pm\sqrt{4e^y + c_1}}{e^y}$. Thus $\dfrac{e^y y'}{\pm\sqrt{4e^y + c_1}} = 1$, and $\dfrac{\pm 1}{2}\sqrt{4e^y + c_1} = t + c_2$. Solving now gives $e^y = (t + c_2)^2 + c_1$, or $y = \ln[(t + c_2)^2 + c_1]$.

41. Let $v = y'$, so $y'' = v(dv/dy)$. Then $v(dv/dy) = 3y^2$, so $\dfrac{1}{2}v^2 = y^3 + c_1$. Now $v^2 = 2y^3 + c_1$ and $v = \pm\sqrt{2y^3 + c_1}$. Since $y' = v$, we have $y' = \pm\sqrt{2y^3 + c_1}$. Since $y'(0) = 4$ and $y(0) = 2$, we have $4 = \pm\sqrt{2(2)^3 + c_1}$, so $c_1 = 0$ (which is a good thing, or we'd be stuck!). Now $y' = \pm\sqrt{2}y^{3/2}$, so $\dfrac{y'}{y^{3/2}} = \pm\sqrt{2}$. Integrating gives $-2y^{-1/2} = \pm\sqrt{2}t + c_2$, so $y = \dfrac{\pm 4}{(\sqrt{2}t + c_2)^2} = \dfrac{2}{(t + c_2)^2}$. Finally (!), $y(0) = 2$ implies that $2 = \dfrac{2}{c_2^2}$, so $c_2 = \pm 1$. To decide which, we differentiate: $y' = \dfrac{-4}{(t \pm 1)^3}$, with $y'(0) = -\dfrac{4}{\pm 1} = 4$, so $c_2 = -1$. Therefore, $y(t) = \dfrac{2}{(t-1)^2}$.

43. Put $v = y', v' = y''$. We have $vv' = t$, so $\dfrac{1}{2}v^2 = \dfrac{1}{2}t^2 + c_1$, or $(y')^2 = t^2 + c_1$. With $y'(1) = 1$, this gives $1^1 = 1^2 + c_1$, so $c_1 = 0$. Now $(y')^2 = t^2$ implies $y' = \pm t$, so $y = \pm\dfrac{1}{2}t^2 + c_2$. With $y(1) = 2$, we get $2 = \dfrac{1}{2} + c_2$, so $c_2 = \dfrac{3}{2}$. Therefore, $y(t) = \dfrac{1}{2}t^2 + \dfrac{3}{2}$.