Solutions to Homework Assignment 11

MATH 256-01
Section 3.2, Page 145
Problems: 1-14, 17, 18, 21-26

1. $W = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{3}{2}e^{t/2} - 2e^{t/2} = -\frac{7}{2}e^{t/2}$.

2. $W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$.

3. $W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix} = e^{-4t}(1 - 2t + 2t) = e^{-4t}$.

4. $W = \begin{vmatrix} x & xe^x \\ 1 & e^x(1+x) \end{vmatrix} = [x(x+1) - x]e^x = x^2e^x$.

5. $W = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = e^{2t}(\sin t \cos t - \sin^2 t - \sin t \cos t - \cos^2 t) = -e^{2t}$.

6. $W = \begin{vmatrix} \cos^2 \theta & 1 + \cos 2\theta \\ -2\cos\theta \sin\theta & -2\sin 2\theta \end{vmatrix} = \begin{vmatrix} \cos^2 \theta & 2\cos^2 \theta \\ -\sin 2\theta & -2\sin 2\theta \end{vmatrix} = 0$.

7. Rewrite as $y'' + \frac{3}{t}y = 1$. Since $\frac{3}{t}$ is continuous on $(-\infty, 0)$ and $(0, \infty)$ and our initial point is $t_0 = 1$, our interval is $(0, \infty)$.

8. $y'' + \frac{-3t}{t-1}y' + \frac{4}{t-1}y = \frac{\sin t}{t-1}$. With the initial point at $t_0 = -2$, our interval is $(-\infty, 1)$.

9. $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$, so our interval is $(0, 4)$. ($t_0 = 3$.)

10. The interval is $(0, \infty)$ since we have $t_0 = 2$.

11. $y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3} = 0$. Since we have $t_0 = 1$, our interval is $(0, 3)$.

12. $y'' + \frac{1}{x-2}y' + (\tan x)y = 0$. We have $t_0 = 3$ and discontinuities at $x = 2$ and odd multiples of $\frac{\pi}{2}$. Our interval is $(2, 3\pi/2)$.

13. $y_1 = t^2, y_1' = 2t$, and $y_1'' = 2$. Thus $t^2y_1'' - 2y_1 = 2t^2 - 2t^2 = 0$. $y_2 = t^{-1}, y_2' = -t^{-2}$, and $y_2'' = 2t^{-3}$. Thus $t^2y_2'' - 2y_2 = 2t^{-1} - 2t^{-1} = 0$. Since the equation is linear, the work we did in class shows that $c_1y_1 + c_2y_2$ is also a solution, but you should verify this directly. I am not going to.

14. $y_1 = 1$ is clearly a solution. $y_2 = t^{1/2}, y_2' = \frac{1}{2}t^{-1/2}$, and $y_2'' = -\frac{1}{4}t^{-3/2}$. Now $y_2y_2'' + (y_2')^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$, so y_2 is a solution. However, if $y = 1 + t^{1/2}$, then $y' = \frac{1}{2}t^{-1/2}$, and $y'' = -\frac{1}{4}t^{-3/2}$, so $yy'' + (y')^2 = -\frac{1}{4}(1 + t^{1/2})(t^{-3/2}) + \frac{1}{4}t^{-1} = -\frac{1}{4}t^{-3/2} \neq 0$. Theorem 3.2.2 only refers to solutions of the linear equation given; this equation is not linear.

17. We know $\begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} = 3e^{4t}$, so $e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. The integrating factor we need is e^{-4t}; multiplying through by this gives $e^{-2t}g'(t) - 2e^{-2t}g(t) = 3$, so $e^{-2t}g(t) = 3t + C$ and $g(t) = 3te^{2t} + Ce^{2t}$.

18. We know $\begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = t^2e^t$, so $tg'(t) - g(t) = t^2e^t$. The integrating factor we need is $\frac{1}{t^2}$; we get $\frac{1}{t}g'(t) - \frac{1}{t^2}g(t) = e^t$, so $\frac{1}{t}g(t) = e^t + C$ and $g(t) = te^t + Ct$.

21. The characteristic equation is $r^2+r-2=0$, with solutions $r=1,-2$. We have $y=c_1e^t+c_2e^{-2t}$. We need $y_1(0)=1$ and $y_1'(0)=0$, so $c_1+c_2=1$ and $c_1-2c_2=0$. Solving gives $c_1=\frac{2}{3}$ and $c_2=\frac{1}{3}$, so $y_1=\frac{1}{3}(2e^t+e^{-2t})$.

For y_2, we need $c_1+c_2=0$ and $c_1-2c_2=1$, so $c_1=\frac{1}{3}$ and $c_2=-\frac{1}{3}$. Thus $y_2=\frac{1}{3}(e^t-e^{-2t})$.

22. $y=c_1e^{-(t-1)}+c_2e^{-3(t-1)}$. [Note: I translated the exponents to make it easier to solve for c_1 and c_2.] For y_1, we have $c_1+c_2=1$ and $-c_1-3c_2=0$, so $c_1=\frac{3}{2}$ and $c_2=-\frac{1}{2}$. Therefore, $y_1=\frac{3}{2}e^{-(t-1)}-\frac{1}{2}e^{-3(t-1)}$.

For y_2, we have $c_1+c_2=0$ and $-c_1-3c_2=1$, so $c_1=\frac{1}{2}$ and $c_2=-\frac{1}{2}$. Therefore, $y_2=\frac{1}{2}e^{-(t-1)}-\frac{1}{2}e^{-3(t-1)}$.

23. $y_1''=-4\cos 2t$, so $y_1''+4y_1=0$. $y_2''=-4\sin 2t$, so $y_2''+4y_2=0$. $\begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}=2\cos^2 2t+2\sin^2 2t=2$, so these form a fundamental set of solutions.

24. $y_1''=y_1'=y_1=e^t$, so $y_1''-2y_1'+y_1=0$. $y_2'=te^t+e^t$ and $y_2''=te^t+2e^t$. Thus $y_2''-2y_2'+y_2=te^t+2e^t-2te^t-2e^t+te^t=0$, so these are both solutions.

$\begin{vmatrix} e^t & te^t \\ e^t & te^t+e^t \end{vmatrix}=te^{2t}+e^{2t}-te^{2t}=e^{2t}\neq 0$, so these form a fundamental set of solutions.

25. $y_1'=1$ and $y_1''=0$, so y_1 is a solution. (Checks easily.) $y_2'=xe^x+e^x$ and $y_2''=xe^x+2e^x$. Now $x^2(xe^x+2e^x)-x(x+2)(xe^x+e^x)+(x+2)xe^x=x^3e^x+2x^2e^x-x^3e^x-x^2e^x-2x^2e^x-2xe^x+x^2e^x+2xe^x=0$, so y_2 is a solution.

$\begin{vmatrix} x & xe^x \\ 1 & xe^x+e^x \end{vmatrix}=x^2e^x+xe^x-xe^x=x^2e^x\neq 0$ for $x>0$, so these form a fundamental set of solutions.

26. Again, $y_1''=0$, so it is easy to check that y_1 is a solution. For y_2, we have $(1-x\cot x)(-\sin x)-x\cos x+\sin x=-\sin x+x\cos x-x\cos x+\sin x=0$.

$\begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}=x\cos x-\sin x$. Since this is not identically zero in $(0,\pi)$, x and $\sin x$ form a fundamental set of solutions on this interval.