Solutions to Homework Assignment 12

MATH 256-01
Section 3.3, Page 152
Problems: 1-10, 15-18, 21-25, 28

1. $W(f,g) = \begin{vmatrix} t^2+5t & t^2-5t \\ 2t+5 & 2t-5 \end{vmatrix} = (2t^3+5t^2-25t)-(2t^3-5t-)25t = 10t$, so these are linearly independent.

2. $\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta \sin\theta = (cos^3\theta - \cos\theta\sin^2\theta) - (2\sin^2\theta\cos\theta) = \cos^3\theta - \cos\theta + \cos^3\theta - 2\cos\theta - 2\cos^2\theta = 4\cos^3\theta - 3\cos\theta$. Since f and g are equal, they are also linearly dependent.

3.

$$\begin{aligned} W(f,g) &= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t) & e^{\lambda t}(\lambda\sin\mu t + \mu\cos\mu t) \end{vmatrix} \\ &= e^{2\lambda t}(\lambda\sin\mu t\cos\mu t + \mu\cos^2\mu t) - e^{2\lambda t}(\lambda\sin\mu t\cos\mu t - \mu\sin^2\mu t) \\ &= e^{2\lambda t}. \end{aligned}$$

Thus, f and g are linearly independent in $\mathbb{R}$.

4. $W(f,g) = \begin{vmatrix} e^{3x} & e^{3(x-1)} \\ 3e^{3x} & 3e^{3(x-1)} \end{vmatrix} = e^{6x}(3e^{-3} - 3e^{-3}) = 0$, so f and g are linearly dependent.

5. $g(t) = 3f(t)$, so $3f(t) + (-1)g(t) = 0$ and they are linearly dependent.

6. $W(f,g) = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = -t^{-1} - t^{-1} = -2t^{-1}$, so f and g are linearly independent.

7. $W(f,g) = \begin{vmatrix} 3t & |t| \\ 3 & \mathrm{sgn}(t) \end{vmatrix} = 3t\mathrm{sgn}(t) - 3|t| = 0$, so f and g are linearly dependent. However, this assumes that 0 is not in the interval since g is not differentiable at 0. If 0 is in the interval, we approach this differently. Suppose $a(3t) + b(|t|) = 0$ for all $t \in I$. We may as well assume that $\pm 1 \in I$. Then $3a + b = 0$ and $-3a + b = 0$, so $a = b = 0$ and f and g are linearly independent.

8. Again, g is not differentiable at $x = 0$, so we attack differently. $ax^3 + b|x|^3 = 0$ for all x implies that $a + b = 0$ and $-a + b = 0$, so $a = b = 0$ (if 0 is internal to the interval). If not, then $W(f,g) = \begin{vmatrix} x^3 & |x|^3 \\ 3x^2 & 3x^2\mathrm{sgn}(x) \end{vmatrix} = 3x^5\mathrm{sgn}(x) - 3|x|^5 = 0$, so f and g are linearly dependent.

9. The functions are independent since the Wronskian is not identically zero.

10. The functions are independent since the Wronskian is not identically zero.

15. $y'' - \dfrac{t+2}{t}y' + \dfrac{t+2}{t^2}y = 0$, so we will need $\displaystyle\int (-\frac{t+2}{t})dt = -t - 2\ln|t| + C$. This gives $W(y_1, y_2) = e^{t+2\ln|t|+C} = Ct^2e^t$.

16. $y'' + \dfrac{\sin t}{\cos t}y' - \dfrac{t}{\cos t}y = 0$. We will need $\displaystyle\int \frac{\sin t}{\cos t}dt = -\ln|\cos t| + C$, so $W(f,g) = e^{\ln|\cos t|+C} = C|\cos t|$. We can absorb the $\pm$ into the C, so we have $C\cos t$. (Note that $p(t)$ is only continuous on $(2k\pi, 2(k+1)\pi)$.)

17. $y'' + \dfrac{1}{x} + \dfrac{x^2 - v^2}{x^2}y = 0$. We have $W(f,g) = e^{-\ln|x|+C} = \dfrac{C}{|x|}$, or just $\dfrac{C}{x}$ since in any interval of solution, $\mathrm{sgn}(x)$ is constant.

18. $y'' - \dfrac{2x}{1-x^2}y' + \dfrac{\alpha(\alpha+1)}{1-x^2}y = 0$, so we will need $\displaystyle\int \frac{-2x}{1-x^2}dx = \ln|1-x^2| + C$, so $W(f,g) = e^{-\ln|1-x^2|+C} = \dfrac{C}{|1-x^2|}$, or just $\dfrac{C}{1-x^2}$.

21. $W = ce^{\int (2/t^2)dt} = ce^{-2/t}$. Since $W(2) = 3$, we have $c/e = 3$, so $c = 3e$. Therefore, $W(4) = 3e^{-2/4+1} = 3\sqrt{e}$.

22. For $W = e^{-\int p(t)dt}$ to be constant, we must have $\displaystyle\int p(t)dt$ constant; that is, independent of t. The only way for this to happen is for $p(t)$ to equal 0 for all t. We cannot conclude anything from this about $q(t)$.

23. $W(fg, fh) = \begin{vmatrix} fg & fh \\ f'g + g'f & f'h + h'f \end{vmatrix} = fgf'h+f^2gh'-fhf'g-f^2hg' = f^2(gh'-g'h) = f^2 \begin{vmatrix} g & h \\ g' & h' \end{vmatrix} = f^2W(f, g)$.

24. If $f(t_0) = g(t_0) = 0$, then $W(t_0) = \begin{vmatrix} 0 & 0 \\ f'(t_0) & g'(t_0) \end{vmatrix} = 0$, so by the remarks on page 151, f and g cannot form a fundamental set of solutions.

25. If y_1 and y_2 have maxima or minima at the same point t_0, then $y_1'(t_0) = y_2'(t_0) = 0$, and again $W(t_0) = 0$.

28. $f'(t) = 3t^2\text{sgn}(t)$ and $g'(t) = 3t^2$, so we have $W(t) = \begin{vmatrix} t^3\text{sgn}(t) & t^3 \\ 3t^2\text{sgn}(t) & 3t^2 \end{vmatrix} = 3t^5\text{sgn}(t) - 3t^5\text{sgn}(t) = 0$ on $(-1, 1)$. Since $f = -g$ on $(-1, 0)$ and $f = g$ on $(0, 1)$, f and g are linearly dependent on each of these intervals. However, they are linearly independent on $(-1, 1)$: If $at^2|t| + bt^3 = 0$ for all $t \in (-1, 1)$, then this must be true for both $t = \pm 1/2$. These give $\frac{1}{8}a - \frac{1}{8}b = 0$ and $\frac{1}{8}a + \frac{1}{8}b = 0$, so $a = b = 0$.