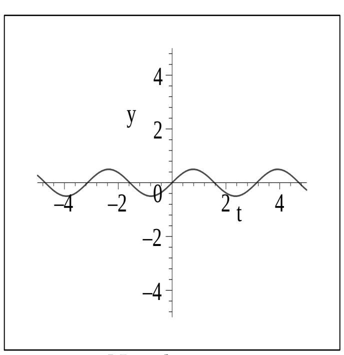
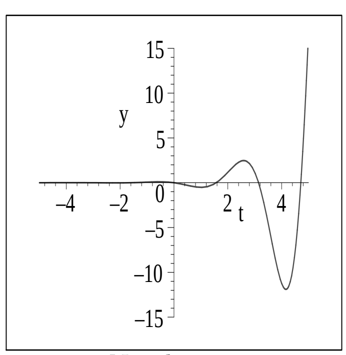
Solutions to Homework Assignment 13

MATH 256-01
Section 3.4, Page 158
Problems: 1-6, 7-21 odd, 24, 28, 29, 33, 35, 36, 37

1. $e^{1+2i} = e(\cos 2 + i \sin 2)$
2. $e^{2-3i} = e^2(\cos(-3) + i\sin(-3)) = e^2(\cos 3 - i \sin 3)$.
3. $e^{i\pi} = \cos\pi + i\sin\pi = -1$.
4. $e^{2-i\pi/2} = e^2(\cos(\pi/2) - i\sin(\pi/2)) = -ie^2$.
5. $2^{1-i} = e^{(i-1)\ln 2} = e^{-\ln 2}(\cos(\ln 2) + i\sin(\ln 2)) = \frac{1}{2}\left(\cos(\ln 2) + i\sin(\ln 2)\right)$.
6. $\pi^{-1+2i} = e^{(-1+2i)\ln\pi} = e^{-\ln\pi}(\cos(2\ln\pi) + i\sin(2\ln\pi)) = \frac{1}{\pi}(\cos(2\ln\pi) + i\sin(2\ln\pi))$.
7. The characteristic equation is $r^2 - 2r + 2 = 0$, so $r = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$. Thus, the general solution is $e^t(c_1\cos t + c_2 \sin t)$.
9. We have $r^2 + 2r - 8 = 0$, so $r = -4$ or $r = 2$. Thus, $y(t) = c_1e^{-4t} + c_2e^{2t}$.
11. We have $r^2 + 6r + 13 = 0$, so $r = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$. Thus, $y(t) = e^{-3t}(c_1\cos 2t + c_2 \sin 2t)$.
13. We have $r^2 + 2r + 1.25 = 0$, so $r = \frac{-2 \pm \sqrt{-1}}{2} = -1 \pm \frac{1}{2}i$. Thus, $y(t) = e^{-t}(c_1\cos(t/2) + c_2\sin(t/2))$.
15. We have $r^2 + r + 1.25 = 0$, so $r = -\frac{1}{2} \pm i$. Thus, $y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t)$.
17. We have $r^2 + 4 = 0$, so $r = \pm 2i$. Thus $y(t) = c_1 \cos 2t + c_2 \sin 2t$. With $y(0) = 0$, we get $c_1 = 0$. With $y'(0) = 1$, we get $2c_2 \cos 2t = 1$, so $c_2 = 1/2$. Therefore, $y(t) = \frac{1}{2}\sin 2t$. Its graph is below; the function just continues to oscillate at the same amplitude as $t \to \infty$.
19. We have $r^2 - 2r + 5 = 0$, so $r = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$. Thus $y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$. With $y(\pi/2) = 0$, we have $c_1 = 0$, so $y(t) = c_2e^t \sin 2t$. This implies $y'(t) = c_2e^t \sin 2t + 2c_2e^t \cos 2t$. With $y'(\pi/2) = 2$, we get $-2c_2e^{\pi/2} = 2$, so $c_2 = -e^{-\pi/2}$. Therefore, $y(t) = -e^{t-\pi/2}\sin 2t$. This is an exponentially growing oscillation.
21. We have $r^2 + r + 1.25 = 0$, so $r = \frac{-1 \pm \sqrt{-4}}{2} = -\frac{1}{2} \pm i$. The general solution is therefore $y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t)$. With $y(0) = 3$, we have $c_1 = 3$, so $y(t) = e^{-t/2}(3\cos t + c_2 \sin t)$. Thus $y'(t) = -\frac{1}{2}e^{-t/2}(3\cos t + c_2 \sin t) + e^{-t/2}(-3\sin t + c_2 \cos t)$. Since $y'(0) = 1$, we get $-\frac{3}{2} + c_2 = 0$, so $c_2 = \frac{3}{2}$. Therefore, $y(t) = e^{-t/2}\left(\cos t - \frac{3}{2}\sin t\right)$. This is an exponentially decaying oscillation.



Number 17



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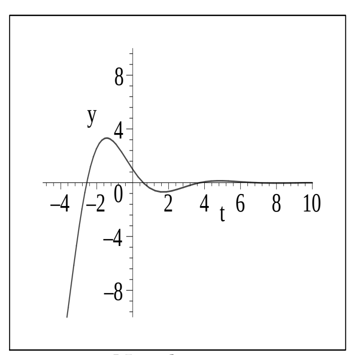

Number 21

24. (a) We have $5r^2+2r+7=0$, so $r = \dfrac{-2 \pm \sqrt{-136}}{10} = \dfrac{-1 \pm i\sqrt{34}}{5}$. Thus $u(t) = e^{-t/5}(c_1\cos(\sqrt{34}t/5) + c_2\sin(\sqrt{34}t/5))$. $u(0) = c_1 = 2$, so $u(t) = e^{-t/5}(2\cos(\sqrt{34}t/5) + c_2\sin(\sqrt{34}t/5))$. $u'(t) = -\dfrac{1}{5}e^{-t/5}((2\cos(\sqrt{34}t/5) + c_2\sin(\sqrt{34}t/5)) + e^{-t/5}\left(-\dfrac{\sqrt{68}}{5}\sin(\sqrt{34}t/5) + \dfrac{\sqrt{34}c_2}{5}\cos(\sqrt{34}t/5)\right)$. $u'(0) = -2/5 + \sqrt{34}c_2/5 = 1$, so $c_2 = 7/\sqrt{34}$. Thus $u(t) = e^{-t/5}\left(2\cos(\sqrt{34}t/5) + \dfrac{7}{\sqrt{34}}\sin(\sqrt{34}t/5)\right)$.

(b) According to MAPLE, $|u(t)| < 0.1$ for all $t > 14.511$ (approximately). (I graphed and zoomed in until I could read off the t-coordinate fairly precisely.)

28. (a) We have already seen both that $\cos t$ and $\sin t$ are solutions of $y'' + y = 0$ and that their Wronskian is nonzero.

(b) If $y = e^{it}$, then $y' = ie^{it}$ and $y'' = -e^{it}$. Thus $y'' + y = -e^{it} + e^{it} = 0$, so $y = e^{it}$ is a solution. Since $c_1\cos t + c_2\sin t$ is the general solution of $y'' + y = 0$, we must have $e^{it} = c_1\cos t + c_2\sin t$.

(c) Now $e^0 = 1 = c_1$.

(d) $ie^{it} = -\sin t + c_2\cos t$. With $t = 0$, we get $i = c_2$.

29. $\dfrac{e^{it} + e^{-t}}{2} = \dfrac{\cos t + i\sin t + \cos t - i\sin t}{2} = \cos t$. $\dfrac{e^{it} - e^{-t}}{2} = \dfrac{\cos t + i\sin t - \cos t + i\sin t}{2} = \sin t$.

33. Let a and b, $a < b$, be consecutive zeroes of y_1, and assume that y_2 is never zero between a and b. Then on the interval (a, b), y_2 is either strictly positive or strictly negative; without loss of generality, we may assume that y_2 is positive on (a, b).

Consider now $y(t) = \dfrac{y_1(t)}{y_2(t)}$. Since $y_2(t)$ is nonzero on (a, b), this function is defined and differentiable on (a, b). Notice that $y'(t) = \dfrac{y_1'(t)y_2(t) - y_2'(t)y_1(t)}{[y_2(t)]^2}$, and the Wronskian of y_2 and y_1 appears in the numerator. Since y_1 and y_2 are linearly independent, this is nonzero on (a, b). However, $y(a) = 0$ and $y(b) = 0$, so by Rolle's theorem, $y'(c) = 0$ for some $c \in (a, b)$.

This means that y_2 must be zero somewhere in (a, b).

We may argue similarly that between consecutive zeroes of y_2 there must be a zero of y_1, so there is one and only one zero of y_2 between consecutive zeroes of y_1.

35. First, $\dfrac{q' + 2pq}{2q^{3/2}} = \dfrac{-2te^{-t^2} + 2te^{-t^2}}{2e^{-3t^2/2}} = 0$, which is constant. Let $x = inte^{-t^2/2}dt$. Then we have $e^{-t^2}\dfrac{d^2y}{dx^2} + \left(-te^{-t^2/2} + te^{-t^2/2}\right)\dfrac{dy}{dx} + e^{-t^2}y = 0$. This simplifies to $\dfrac{d^2y}{dx^2} + y = 0$, which has solution $y = c_1\cos x + c_2\sin x$, where $x = \int e^{-t^2/2}dt$.

36. First, $\dfrac{q' + 2pq}{2q^{3/2}} = \dfrac{3 + 6t^3}{2t^3}$, which is not constant. We will not be able to use this technique on this DE.

37. Rewrite: $y'' + \dfrac{t^2 - 1}{t}y' + t^2y = 0$. Then $\dfrac{q' + 2pq}{2q^{3/2}} = \dfrac{2t + 2(t)(t^2 - 1)}{2t^3} = 1$, so this will work.

Let $x = \int t\,dt = \dfrac{1}{2}t^2$. Then $t^2\dfrac{d^2y}{dt^2} + (1 + (t^2 - 1))\dfrac{dy}{dx} + t^2y = 0$ gives $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$, so we get the characteristic equation $r^2 + r + 1 = 0$. Our roots are $r = \dfrac{-1 \pm i\sqrt{3}}{2}$. The solution is therefore $e^{-x/2}(c_1\cos(\sqrt{3}x/2) + c_2\sin(\sqrt{3}x/2)) = e^{-t^2/4}(c_1\cos(\sqrt{3}t^2/4) + c_2\sin(\sqrt{3}t^2/4))$.