Solutions to Homework Assignment 14

MATH 256-01
Section 3.5, Page 166
Problems: 1-13 odd, 16, 20, 23-29 odd

1. The characteristic equation is $r^2 - 2r + 1 = 0$, so $r = 1$ is a repeated root. The general solution is therefore $y(t) = c_1e^t + c_2te^t$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, so the roots are $r = 3/2$ and $r = -1/2$. The general solution is therefore $y(t) = c_1e^{-t/2} + c_2e^{3t/2}$.

5. The characteristic equation is $r^2 - 2r + 10 = 0$, so the roots are $r = \dfrac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$. The general solution is therefore, $y(t) = e^t(c_1 \cos 3t + c_2 \sin 3t)$.

7. The characteristic equation is $4r^2 + 17r + 4 = 0$, so the roots are $r = -4$ and $r = -1/4$. The general solution is therefore $y(t) = c_1e^{-4t} + c_2e^{-t/4}$.

9. The characteristic equation is $25r^2 - 20r + 4 = 0$, so the root $r = 2/5$ is a repeated root. The general solution is therefore $y(t) = c_1e^{2t/5} + c_2te^{2t/5}$.

11. The characteristic equation is $9r^2 - 12r + 4 = 0$, so the root $r = 2/3$ is a repeated root. The general solution is therefore $y(t) = c_1e^{2t/3} + c_2te^{2t/3}$. Since $y(0) = 2$, $c_1 = 2$. $y'(t) = \dfrac{4}{3}e^{2t/3} + c_2e^{2t/3} + \dfrac{2c_2t}{3}e^{2t/3}$. With $y'(0) = -1$, this is $-1 = \dfrac{4}{3} + c_2$, so $c_2 = -\dfrac{7}{3}$. Thus $y(t) = 2e^{2t/3} - \dfrac{7t}{3}e^{2t/3}$. As $t \to \infty$, $y \to -\infty$.

13. The characteristic equation is $9r^2 + 6r + 82 = 0$, so the roots are $r = \dfrac{-6 \pm \sqrt{-2916}}{18} = -\dfrac{1}{3} \pm 3i$. The general solution is therefore $y(t) = e^{-t/3}(c_1 \cos 3t + c_2 \sin 3t)$. Since $y(0) = -1$, we have $c_1 = -1$. Now $y'(t) = -\dfrac{1}{3}e^{-t/3}(-\cos 3t + c_2 \sin 3t) + e^{-t/3}(-3 \sin 3t + 3c_2 \cos 3t)$. $y'(0) = 2$ gives $-\dfrac{1}{3}(-1) + (3c_2) = 2$, so $c_2 = \dfrac{5}{9}$. Thus $y(t) = e^{-t/3}\left(-\cos 3t + \dfrac{5}{9}\sin 3t\right)$. We have $y \to 0$ as $t \to \infty$.

16. The general solution (see page 164) is $y(t) = c_1e^{t/2} + c_2te^{t/2}$. Since $y(0) = 2$, $c_1 = 2$. Now $y'(t) = e^{t/2} + c_2e^{t/2} + \dfrac{c_2t}{2}e^{t/2}$. Since $y'(0) = b$, we have $1 + c_2 = b$, and $c_2 = b - 1$. The solution is therefore $y(t) = 2e^{t/2} + (b-1)te^{t/2}$. If $b \geq 1$, the solutions will grow positively; if $b < 1$, they will grow negatively.

20. (a) This is simple algebra.
    (b) $W = c_1e^{-\int 2a dt} = c_1e^{-2at}$.
    (c) We know that $y_1 = e^{-at}$ is a solution and that $W = y_1y_2' - y_2y_1'$. Thus $e^{-at}y_2' + ae^{-at}y_2 = c_1e^{-2at}$. We get $e^{at}y_2' + ae^{at}y_2 = c_1$. Now integrate both sides with respect to t: $e^{at}y_2 = c_1t + c_2$, and $y_2 = c_1te^{-at} + c_2e^{-at}$. This gives us te^{-at} as another solution.

23. Let $y = vt^2$. Then $y' = v't^2 + 2vt$ and $y'' = v''t^2 + 4v't + 2v$. We get $t^2(v''t^2 + 4v't + 2v) - 4t(v't^2 + 2vt) + 6vt^2 = 0$, so $t^4v'' = 0$, so $v'' = 0$. Thus $v = c_1t + c_2$, and $y = t^3$ is another solution.

25. Let $y = vt^{-1}$. Then $y' = v't^{-1} - vt^{-2}$ and $y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$. We get $t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) + vt^{-1} = tv'' + v' = 0$. With $u = v'$, this is $tu' + u = 0$, so $tu = c_1$ and $u = \dfrac{c_1}{t}$. Thus $v = c_1 \ln t + c_2$. Our new solution is thus $t^{-1}\ln t$. (That's vt^{-1}.)

27. Let $y = v \sin x^2$. Then $y' = v' \sin x^2 + 2xv \cos x^2$ and $y'' = v'' \sin x^2 + 4xv' \cos x^2 - 4x^2v \sin x^2 + 2v \cos x^2$. We get $x(v'' \sin x^2 + 4xv' \cos x^2 - 4x^2v \sin x^2 + 2v \cos x^2) - (v' \sin x^2 + 2xv \cos x^2) + 4x^3v \sin x^2 = x \sin x^2 v'' + (4x^2 \cos x^2 - \sin x^2)v' = 0$. Let $u = v'$. The resulting equation is separable: $\dfrac{u'}{u} = \dfrac{1}{x} -$

$\dfrac{4x\cos x^2}{\sin x^2}$. Integration gives $lnu = \ln x - 2\ln(\sin x^2) + C$, so $u = v' = \dfrac{c_1 x}{\sin^2 x^2} = c_1 x \csc^2(x^2)$. Recall that the derivative of $\cot z$ is $-csc^2 z$, so we have $v = -\dfrac{1}{2}c_1 \cot(x^2)$.

Finally, $y = v\sin x^2 = -\dfrac{1}{2}c_1\cos x^2$, so we may use $y = \cos x^2$ as our other solution.

29. Let $y = vx^{1/4}e^{2\sqrt{x}}$. Then $y' = v'x^{1/4}e^{2\sqrt{x}} + \frac{1}{4}vx^{-3/4}e^{2\sqrt{x}} + vx^{-1/4}e^{2\sqrt{x}} = e^{2\sqrt{x}}\left(v'x^{1/4} + \frac{1}{4}vx^{-3/4} + vx^{-1/4}\right)$ and

$$\begin{aligned} y'' &= \frac{e^{2\sqrt{x}}}{x^{1/2}}\left(v'x^{1/4} + \frac{1}{4}vx^{-3/4} + vx^{-1/4}\right) + e^{2\sqrt{x}}\left(v''x^{1/4} + \frac{1}{2}v'x^{-3/4} - \frac{3}{16}vx^{-7/4} + v'x^{-1/4} - \frac{1}{4}vx^{-5/4}\right) \\ &= e^{2\sqrt{x}}\left[v''x^{1/4} + v'\left(2x^{-1/4} + \frac{1}{2}x^{-3/4}\right) + v\left(x^{-3/4} - \frac{3}{16}x^{-7/4}\right)\right]. \end{aligned}$$

Now

$$\begin{aligned} x^2y'' - (x - 3/16)y &= e^{2\sqrt{x}}\left[v''x^{9/4} + v'\left(2x^{7/4} + \frac{1}{2}x^{5/4}\right) + v\left(x^{5/4} - \frac{3}{16}x^{1/4}\right)\right] - (x - 3/16)vx^{1/4}e^{2\sqrt{x}} \\ &= e^{2\sqrt{x}}\left[v''x^{9/4} + v'\left(2x^{7/4} + \frac{1}{2}x^{5/4}\right)\right] \\ &= 0. \end{aligned}$$

Thus, our differential equation is now the first-order equation (in v')

$$v''x^{9/4} + v'\left(2x^{7/4} + \frac{1}{2}x^{5/4}\right) = 0.$$

This is separable: $\dfrac{v''}{v'} = -2x^{-1/2} - \dfrac{1}{2}x^{-1}$. Thus $\ln|v'| = -4x^{1/2} - \dfrac{1}{2}\ln x + c_1$, so $v' = \pm\dfrac{c_1 e^{-4\sqrt{x}}}{\sqrt{x}}$, so $v = \mp\dfrac{c_1}{2}e^{-4\sqrt{x}} + c_2$. For our solution y, we may ignore the constants; we get $y = vx^{1/4}e^{2\sqrt{x}} = x^{1/4}e^{-2\sqrt{x}}$.