

# Solutions to Homework Assignment 17

MATH 256-01

Section 3.6, Page 178

Problems: 1-17 odd, 19-25 odd (a) only

1.  $(D^2 - 2D - 3)y = 3e^{2t}$ , so  $(D - 2)(D - 3)(D + 1)y = (D - 2)(3e^{2t}) = 0$ . The general solution of this homogeneous equation is  $y(t) = c_1e^{2t} + c_2e^{3t} + c_3e^{-t}$ . The last two terms are solutions of the corresponding homogeneous equation, so our particular solution is  $y = c_1e^{2t}$ . We have  $y' = 2c_1e^{2t}$  and  $y'' = 4c_1e^{2t}$ , giving  $c_1e^{2t}(4 - 2(2) - 3) = 3e^{2t}$ , so  $c_1 = -1$ , making the particular solution  $y = -e^{2t}$ . Therefore, the general solution is  $y(t) = -e^{2t} + c_2e^{3t} + c_3e^{-t}$ .

3. We have  $(D + 1)^2(D - 3)(D + 1)y = 0$ , or  $(D + 1)^3(D - 3) = 0$ . The general solution of this homogeneous equation is  $y(t) = c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t} + c_4e^{3t}$ . The first and fourth terms are solutions of the corresponding homogeneous equation, so our particular solution is  $c_2te^{-t} + c_3t^2e^{-t} = (c_2t + c_3t^2)e^{-t}$ . We have  $y' = (c_2 + 2c_3t)e^{-t} - (c_2t + c_3t^2)e^{-t} = (c_2 + (2c_3 - c_2)t - c_3t^2)e^{-t}$  and  $y'' = (2c_3 - c_2 - 2c_3t)e^{-t} - (c_2 + (2c_3 - c_2)t - c_3t^2)e^{-t} = (2c_3 - 2c_2 + (c_2 - 4c_3)t + c_3t^2)e^{-t}$ .

Now  $(2c_3 - 2c_2 + (c_2 - 4c_3)t + c_3t^2)e^{-t} - 2(c_2 + (2c_3 - c_2)t - c_3t^2)e^{-t} - 3(c_2t + c_3t^2)e^{-t} = -3te^{-t}$ .

This implies  $2c_3 - 4c_2 = 0$  (no  $e^{-t}$  terms) and  $-8c_3 = -3$ . Thus  $c_3 = \frac{3}{8}$  and  $c_2 = \frac{3}{16}$ . The general solution is therefore  $y(t) = c_1e^{-t} + \frac{3}{16}te^{-t} + \frac{3}{8}t^2e^{-t} + c_4e^{3t}$ .

5. The annihilator is  $D(D - 3)^3$ , so the differential equation is  $D(D - 3)^3(D^2 + 9)y = 0$ . The particular solution will have the form  $y(t) = c_1 + c_2e^{3t} + c_3te^{3t} + c_4t^2e^{3t}$ . Ugh.

$y' = (3c_2 + c_3 + 3c_3t + 2c_4t + 3c_4t^2)e^{3t}$  and  $y'' = (3c_3 + 2c_4 + 6c_4t + 9c_2 + 3c_3 + 9c_3t + 6c_4t + 9c_4t^2)e^{3t}$ . We get

$$9(3c_3 + 2c_4 + 6c_4t + 9c_2 + 3c_3 + 9c_3t + 6c_4t + 9c_4t^2) + 9(c_2 + c_3t + c_4t^2)]e^{3t} + 9c_1 = t^2e^{3t} + 6 \\ 9c_1 + (2c_4 + 18c_2 + 6c_3) + t(12c_4 + 18c_3) + t^2(18c_4) = t^2e^{3t} + 6.$$

This gives  $9c_1 = 6$  and  $c_4 = 1/18$ . From there, we find  $c_3 = -1/27$  and  $c_2 = 1/162$ . The general solution is  $y(t) = c_1 \cos t + c_2 \sin t + \frac{2}{3} + \frac{1}{162}e^{3t} - \frac{1}{27}te^{3t} + \frac{1}{18}t^2e^{3t}$ .

7. We have  $D^3(D^2 + 1)(2D + 1)(D + 1)y = 0$ . The particular solution has the form  $y(t) = c_1 + c_2t + c_3t^2 + c_4 \cos t + c_5 \sin t$ . (We could also solve for  $t^2$  and  $3 \sin t$  separately and add the solutions.)

$y'(t) = c_2 + 2c_3t - c_4 \sin t + c_5 \cos t$  and  $y''(t) = 2c_3 - c_4 \cos t - c_5 \sin t$ . We get

$$2(2c_3 - c_4 \cos t - c_5 \sin t) + 3(c_2 + 2c_3t - c_4 \sin t + c_5 \cos t) + (c_1 + c_2t + c_3t^2 + c_4 \cos t + c_5 \sin t) = t^2 + 3 \sin t.$$

Now  $4c_3 + 3c_2 + c_1 = 0$ ,  $6c_3 + c_2 = 0$ ,  $c_3 = 1$  gives  $c_2 = -6$  and  $c_1 = 14$ . Also,  $(-c_4 + 3c_5) = 0$  and  $(-c_5 - 3c_4) = 3$  gives  $10c_5 = -9$ , so  $c_5 = -3/10$  and  $c_4 = -9/10$ . The general solution is therefore

$$y(t) = c_1e^{-t/2} + c_2e^{-t} + t^2 - 6t + 14 - \frac{9}{10} \cos t - \frac{3}{10} \sin t.$$

9. The annihilator is  $D^2 + \omega^2$ , and it does not reproduce any roots of the original characteristic equation. The form of the particular solution is  $c_1 \cos \omega t + c_2 \sin \omega t$ . We get  $-c_1\omega^2 \cos \omega t - c_2\omega^2 \sin \omega t + \omega_0^2(c_1 \cos \omega t + c_2 \sin \omega t) = \cos \omega t$ .

Thus  $-c_2\omega^2 + c_2\omega_0^2 = 0$ , so  $c_2 = 0$ , and  $-c_1\omega^2 + c_1\omega_0^2 = 1$ , so  $c_1 = \frac{1}{\omega_0^2 - \omega^2}$ . The general solution is

$$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{1}{\omega_0^2 - \omega^2} \cos \omega t.$$

11.  $(D^2 - 1)$  will annihilate  $\sinh t$  since  $D(\sinh t) = \cosh t$  and  $D(\cosh t) = \sinh t$ . This will not duplicate any solutions of the homogeneous equation, so we get the particular solution  $y = c_1e^t + c_2e^{-t}$ , which

we could (and probably should) rewrite as  $Y = c_1 \cosh t + c_2 \sinh t$  (with a different choice of  $c_1$  and  $c_2$ ). We get

$(c_1 \cosh t + c_2 \sinh t) + (c_1 \sinh t + c_2 \cosh t) + 4(c_1 \cosh t + c_2 \sinh t) = 2 \sinh t$ , so  $5c_1 + c_2 = 0$  and  $c_1 + 5c_2 = 2$ . This gives  $-24c_2 = -10$ , so  $c_2 = 5/12$ . We also get  $c_1 = -1/12$ . We have  $Y(t) = -\frac{1}{12} \cosh t + \frac{5}{12} \sinh t$ . Solving the corresponding homogenous equation gives us  $r = \frac{-1 \pm \sqrt{-15}}{2}$ , so we have  $y(t) = c_1 e^{-t/2} \cos(\sqrt{15}t/2) + c_2 e^{-t/2} \sin(\sqrt{15}t/2) - \frac{1}{12} \cosh t + \frac{5}{12} \sinh t$ .

13. Our annihilator is  $D^2$ , so we have  $D^2(D+2)(D-1)y = 0$ . We get  $Y(t) = c_1 + c_2 t$ , so  $Y' = c_2$  and  $Y'' = 0$ , leaving us with  $c_2 - 2(c_1 + c_2 t) = 2t$ ; thus,  $c_2 = -1$  and  $c_1 = -\frac{1}{2}$ . Now  $y(t) = c_1 e^{-2t} + c_2 e^t - t - \frac{1}{2}$ .

$y(0) = 0$  gives us  $c_1 + c_2 - \frac{1}{2} = 0$  and  $y'(0) = 1$  gives us  $-2c_1 + c_2 - 1 = 1$ . Solving yields  $c_2 = 1$  and  $c_1 = -\frac{1}{2}$ . Thus  $y(t) = -\frac{1}{2} e^{-2t} + e^t - t - \frac{1}{2}$ .

15. The annihilator is  $D(D-1)^2$ , so we get  $D(D-1)^4 y = 0$ , and  $Y(t) = c_1 + c_2 t^2 e^t + c_3 t^3 e^t$ .  $Y' = e^t(2c_2 + 3c_3 t^2 + c_2 t^2 + c_3 t^3)$  and  $Y'' = e^t(2c_2 + 3c_3 t^2 + c_2 t^2 + c_3 t^3 + 6c_3 t + 2c_2 t + 3c_3 t^2)$ . Now  $e^t(2c_2 + 3c_3 t^2 + c_2 t^2 + c_3 t^3 + 6c_3 t + 2c_2 t + 3c_3 t^2) - 2e^t(2c_2 + 3c_3 t^2 + c_2 t^2 + c_3 t^3) + (c_1 + c_2 t^2 e^t + c_3 t^3 e^t) = te^t + 4$ , so  $c_1 = 4$ ,  $-2c_2 = 0$ , and  $6c_3 = 1$ . Thus  $Y(t) = \frac{1}{6} t^3 e^t + 4$ . The solution to the corresponding homogeneous equation is  $c_1 e^t + c_2 t e^t$ , so  $y(t) = c_1 e^t + c_2 t e^t + \frac{1}{6} t^3 e^t + 4$ .  $y(0) = 1$  gives  $c_1 + 4 = 1$ , so  $c_1 = -3$ .  $y'(0) = 1$  gives  $c_1 + c_2 = 1$ , so  $c_2 = 4$ . Finally,  $y(t) = -3e^t + 4te^t + \frac{1}{6}t^3 e^t + 4$ .

17. The annihilator is  $D^2 + 4$ , so we have  $(D^2 + 4)^2 y = 0$ . The particular solution is  $Y(t) = At \cos 2t + Bt \sin 2t$ . We get  $Y' = A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t$  and  $Y'' = -2A \sin 2t - 2A \sin 2t - 4At \cos 2t + 2B \sin 2t + 2B \cos 2t - 4Bt \sin 2t$ . This gives

$$(-4A \sin 2t - 4At \cos 2t + 2B \sin 2t + 2B \cos 2t - 4Bt \sin 2t) + 4(At \cos 2t + Bt \sin 2t) = 3 \sin 2t,$$

so  $-4A + 2B = 3$  and  $2B = 0$ . This means  $A = -3/4$  and  $B = 0$ , so we now have  $y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{4} t \cos t$ . With  $y(0) = 2$ , we get  $c_1 = 2$ . With  $y'(0) = -1$ , we get  $2c_2 - \frac{3}{4} = -1$ , and  $c_2 = -\frac{1}{8}$ .

Therefore,  $y(t) = 2 \cos 2t - \frac{1}{8} \sin 2t - \frac{3}{4} t \cos t$ .

19. The annihilator is  $D^5(D+3)^3(D^2+3)$ , so the equation becomes  $D^6(D+3)^4(D^2+9)y = 0$ . The particular solution has the form  $c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t e^{-3t} + c_7 t^2 e^{-3t} + c_8 t^3 e^{-3t} + c_9 \cos 3t + c_{10} \sin 3t$ .

21.  $e^t \cos 2t$  is annihilated by  $D^2 - 2D + 5$ .  $e^{2t}(3t+4) \sin t$  is annihilated by  $(D^2 - 4t + 5)^2$ . The original operator is  $(D-2)(D-3)$ . The form of a particular solution is therefore  $Y(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t + c_3 e^{2t} \cos t + c_4 e^{2t} \sin t + c_5 t e^{2t} \cos t + c_6 t e^{2t} \sin t$ .

23. To annihilate these, we need  $D^3, (D-2)^2, (D^2+4)^2$ . The corresponding homogeneous equation is  $(D-2)^2 y = 0$ . Thus, the form of  $Y$  is  $Y(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^2 e^{2t} + c_5 t^3 e^{2t} + c_6 \cos 2t + c_7 \sin 2t + c_8 t \cos 2t + c_9 \sin 2t$ .

25. Ugh. The corresponding homogeneous equation is  $(D+2)(D+1) = 0$ , and the annihilators we need are  $(D^2 - 2D + 5)^3, D^2 + 2D + 2$ , and  $D - 1$ . The form of  $Y$  is therefore

$$Y(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t + c_3 t e^t \cos 2t + c_4 t e^t \sin 2t + c_5 t^2 e^t \cos 2t + c_6 t^2 e^t \sin 2t + c_7 e^{-t} \cos t + c_8 e^{-t} \sin t + c_9 e^t.$$