Solutions to Homework Assignment 18

MATH 256-01
Section 3.7, Page 183
Problems: 1-19 odd

1. The solution to the corresponding homogeneous equation is $c_1e^{2t} + c_2e^{3t}$, so let $Y(t) = u_1e^{2t} + u_2e^{3t}$. Then $Y' = u_1'e^{2t}+2u_1e^{2t}+u_2'e^{3t}+3u_2e^{3t} = 2u_1e^{2t}+3u_2e^{3t}$ once we impose the condition $u_1'e^{2t}+u_2'e^{3t} = 0$.

   Now $Y'' = 2u_1'e^{2t} + 4u_1e^{2t} + 3u_2'e^{3t} + 9u_2e^{3t}$. We get $(2u_1'e^{2t}+4u_1e^{2t}+3u_2'e^{3t}+9u_2e^{3t})-5(2u_1e^{2t}+3u_2e^{3t})+6(u_1e^{2t}+u_2e^{3t}) = 2u_1'e^{2t}+3u_2'e^{3t} = 2e^t$. From before, we also have $u_1'e^{2t} + u_2'e^{3t} = 0$. Solving yields $u_2'e^{3t} = 2e^t$, so $u_2' = 2e^{-2t}$ and $u_2 = -e^{-2t}$. (We do not need the constant of integration since it would just give us a term that solves the homogeneous equation.)

   This gives $u_1'e^{2t} + 2e^{-2t}e^{3t} = 0$ so $u_1' = -2e^{-t}$. Finally, we get $u_1 = 2e^{-t}$, so $Y(t) = 2e^t - e^t = e^t$.

   I will let you solve by undetermined coefficients; the annihilator is $D - 1$.

3. The general solution to the corresponding homogeneous equation is $c_1e^{-t} + c_2te^{-t}$. The Wronskian is $W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}(1-t) \end{vmatrix} = e^{-2t}(1-t) + te^{-2t} = e^{-2t}$. Using the formula on page 183, we get

$$\begin{aligned} Y(t) &= -e^{-t}\int \frac{te^{-t}(3e^{-t})}{e^{-2t}}dt + te^{-t}\int \frac{e^{-t}(3e^{-t})}{e^{-2t}}dt \\ &= -e^{-t}\int 3tdt + te^{-t}\int 3dt \\ &= -e^{-t}\left(\frac{3}{2}t^2\right) + 3t^2e^{-t} \\ &= \frac{3}{2}t^2e^{-t}. \end{aligned}$$

   Again, you can check undetermined coefficients on your own.

5. $y_1 = \cos t$ and $y_2 = \sin t$ solve the corresponding homogeneous equation, and their Wronskian is 1. Thus

$$\begin{aligned} Y(t) &= -\cos t\int \sin t\tan tdt + \sin t\int \cos t\tan tdt \\ &= -\cos t\int \frac{\sin^2 t}{\cos t}dt + \sin t\int \sin tdt \\ &= -\cos t\int \frac{1-\cos^2 t}{\cos t}dt - \sin t\cos t \\ &= -\cos t\int(\sec t - \cos t)dt - \sin t\cos t \\ &= -\cos t(\ln|\sec t + \tan t| - \sin t) - \sin t\cos t \\ &= -\cos t\ln|\sec t + \tan t|. \end{aligned}$$

   Therefore, $y(t) = c_1\cos t+c_2\sin t-\cos t\ln(\sec t+\tan t)$. ($\sin t$ and $\cos t$ are both positive in the interval specified.)

7. The general solution of the corresponding homogeneous equation is $c_1e^{-2t} + c_2te^{-2t}$. The Wronskian

is $W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix} = e^{-4t}[(1-2t)+2t] = e^{-4t}$. Thus

$$\begin{aligned} Y(t) &= -e^{-2t}\int \frac{te^{-2t}(t^{-2}e^{-2t})}{e^{-4t}}dt + te^{-2t}\int \frac{e^{-2t}(t^{-2}e^{-2t})}{e^{-4t}}dt \\ &= -e^{-2t}\int \frac{1}{t}dt + te^{-2t}\int \frac{1}{t^2}dt \\ &= e^{-2t}\ln t - e^{-2t}. \end{aligned}$$

The second term is a solution of the homogeneous equation, so we may disregard it.
The general solution is therefore $y(t) = c_1e^{-2t} + c_2te^{-2t} + e^{-2t}\ln t$.

9. We must first rewrite the equation as $y'' + \frac{1}{4}y = \frac{1}{2}\sec(t/2)$. Now $y_1 = \cos(t/2)$ and $y_2 = \sin(t/2)$ solve the corresponding homogeneous equation, and their Wronskian is $1/2$. Thus

$$\begin{aligned} Y(t) &= -\cos(t/2)\int 2\sin(t/2)(\sec(t/2)/2)dt + \sin(t/2)\int 2\cos(t/2)(\sec(t/2)/2)dt \\ &= 2\cos(t/2)\ln(\cos(t/2)) + \sin(t/2)t. \end{aligned}$$

The general solution is therefore $c_1\cos(t/2) + c_2\sin(t/2) + 2\cos(t/2)\ln|\cos(t/2)| + t\sin(t/2)$.

11. $y_1 = e^{2t}$ and $y_2 = e^{3t}$ solve the corresponding homogeneous equation, and their Wronskian is e^{5t}. Thus

$$\begin{aligned} Y(t) &= -e^{2t}\int e^{-5t}(e^{3t})g(t)dt + e^{3t}\int e^{-5t}(e^{2t})g(t)dt \\ &= -e^{-2t}\int g(t)e^{-2t}dt + e^{3t}\int g(t)e^{-3t}dt. \end{aligned}$$

The general solution is $y(t) = c_1e^{2t} + c_2e^{3t} - e^{-2t}\int g(t)e^{-2t}dt + e^{3t}\int g(t)e^{-3t}dt$.

13. It is easy to verify that y_1 and y_2 are solutions of the corresponding homogeneous equation. Now rewrite the equation as $y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2}$ so it is in the proper form to use Theorem 3.7.1. The Wronskian of t^2 and t^{-1} is -3. We get

$$\begin{aligned} Y(t) &= -t^2\int [(-1/3)t^{-1}(3-t^{-2})]dt + t^{-1}\int [(-1/3)t^2(3-t^{-2})]dt \\ &= -t^2\int (-t^{-1}+t^{-3}/3)dt + t^{-1}\int (-t^2+1/3)dt \\ &= -t^2\left(-\ln t - \frac{1}{6}t^{-2}\right) + t^{-1}(-t^3/3 + t/3) \\ &= t^2\ln t + \frac{1}{6} - t^2/3 + 1/3 \\ &= t^2\ln t - t^2/3 + \frac{1}{2}. \end{aligned}$$

We may ignore the $-t^2/3$ since it is a solution of the corresponding homogeneous equation; we get $Y(t) = t^2\ln t + \frac{1}{2}$.

15. Again, I will leave verification to you. We have $y''\frac{1+t}{t}y' + \frac{1}{t}y = te^{2t}$. The Wronskian of y_1 and y_2 is te^t. Now

$$
\begin{aligned}
Y(t) &= -(1+t)\int \frac{e^t(te^{2t})}{te^t}dt + e^t\int \frac{(t+1)(te^{2t})}{te^t}dt \\
&= -(1+t)\int e^{2t}dt + e^t\int (t+1)e^t dt \\
&= -(1+t)e^{2t}/2 + e^t(te^t) \\
&= \frac{1}{2}(t-1)e^{2t}.
\end{aligned}
$$

17. The Wronskian of y_1 and y_2 is x^3. Rewrite the equation as $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln(x)$. We get

$$
\begin{aligned}
Y(t) &= -x^2\int \frac{x^2\ln(x)(\ln(x))}{x^3}dx + x^2\ln(x)\int \frac{x^2(\ln(x))}{x^3}dx \\
&= -x^2\int \frac{(\ln(x))^2}{x}dx + x^2\ln(x)\int \frac{\ln(x)}{x}dx \\
&= -x^2\cdot\frac{(\ln(x))^3}{3} + x^2\ln(x)\cdot\frac{(\ln(x))^2}{2} \\
&= \frac{1}{6}x^2(\ln(x))^3.
\end{aligned}
$$

19. The Wronskian is $(1-x)e^x$. Rewrite the equation as $y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = \frac{g(x)}{1-x}$. We get

$$
\begin{aligned}
Y(t) &= -e^x\int \frac{xg(x)}{(1-x)e^x}dx + x\int \frac{e^x g(x)}{(1-x)e^x}dx \\
&= -e^x\int \frac{xg(x)}{(1-x)^2e^x}dx + x\int \frac{g(x)}{(1-x)^2}dx.
\end{aligned}
$$