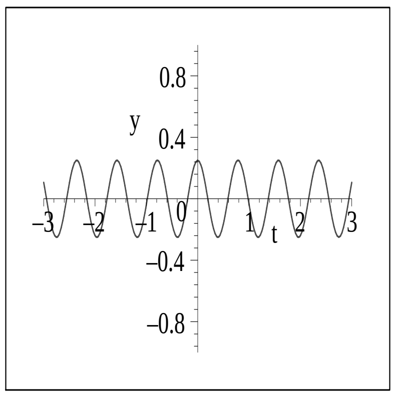
Solutions to Homework Assignment 21

MATH 256-01
Section 3.78, Page 197
Problems: 1, 3, 5, 7, 11, 12, 16, 24

1. $R^2 = A^2 + B^2 = 3^2 + 4^2 = 25$, so $R = 5$. Also, $\omega_0 = 2$ and $\tan\delta = \dfrac{B}{A} = \dfrac{4}{3}$. Thus $u = 5\cos(2t - \delta)$, where $\delta = \arctan(4/3)$.

3. $R^2 = A^2 + B^2 = 16 + 4 = 20$, so $R = 2\sqrt{5}$. We have $\omega_0 = 3$ and $\tan\delta = -\dfrac{1}{2}$. Therefore, $u = 2\sqrt{5}\cos(3t - \delta)$, where $\delta = -\arctan(1/2)$.

5. We have $k = 4$ lb/ft. Since there is no damping, $\gamma = 0$. Our equation is therefore $\dfrac{2}{32}y'' + 4y = 0$, or $y'' + 64y = 0$. The general solution is $y(t) = c_1\cos(8t) + c_2\sin(8t)$. We are told that $y(0) = \dfrac{1}{4}$ and $y'(0) = 0$, so $c_1 = \dfrac{1}{4}$ and $8c_2 = 0$. Thus $c_2 = 0$, and $y(t) = \dfrac{1}{4}\cos(8t)$. We have $\omega_0 = 8$, $T = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$, and $R = \dfrac{1}{4}$.



7. We have $k = 12$ lb/ft and $m = \dfrac{3}{32}$ lb. Also, $y(0) = -1/12$ and $y'(0) = 2$. Since there is no damping, $\gamma = 0$ and our equation is $\dfrac{3}{32}y'' + 12y = 0$, or $y'' + 128y = 0$. Thus the general solution is $y(t) = c_1\cos(8\sqrt{2}t) + c_2\sin(8\sqrt{2}t)$. $y(0) = c_1 = -1/12$ and $y'(0) = 8\sqrt{2}c_2 = 2$, so $c_2 = \dfrac{\sqrt{2}}{8}$.
$y(t) = -\dfrac{1}{12}\cos(8\sqrt{2}t) + \dfrac{\sqrt{2}}{8}\sin(8\sqrt{2}t)$. This has $\omega_0 = 8\sqrt{2}$ and $T = \dfrac{2\pi}{8\sqrt{2}} = \dfrac{\pi}{4\sqrt{2}}$. $R = \sqrt{\dfrac{1}{144} + \dfrac{1}{32}} = \sqrt{\dfrac{11}{288}}$. $\tan\delta = \dfrac{\sqrt{2}/8}{-1/12}$, so $\delta \approx -1.13$ radians (or $\pi - 1.13 \approx 2.01$).

11. We have $m = 2$, $k = 30$ N/m and $\gamma = 0.6$ N/(m/s), so the equation is $2y'' + 0.6y' + 30y = 0$, with $y(0) = 0.05$ and $y'(0) = 0.1$. We need $2r^2 + 0.6r + 30 = 0$, so $r = \dfrac{-0.6 \pm \sqrt{0.36 - 240}}{4} \approx -0.15 \pm i3.87$.
The general solution is therefore $y(t) = c_1e^{-0.15t}\cos(3.87t) + c_2e^{-0.15t}\sin(3.87t)$.
$y(0) = 0.05$, so $c_1 = 0.05$. $y'(0) = c_1(-0.15) + c_2(3.87) = 0.1$, giving $c_2 \approx 0.02\overline{7}$. The amplitude is $\sqrt{(0.05)^2 + (0.02778)^2} \approx 0.0572$. $\mu \approx 3.87$. $\tan\delta \approx \dfrac{0.02778}{0.05}$, so $\delta \approx 0.507$ radians. Whew! We get

$$y(t) = 0.0572\cos(3.87t - 0.507).$$

The quasi-frequency is $\mu = 3.87$. Without damping, we would have found $\omega_0 = \sqrt{15}$, so the ratio is $\dfrac{3.87}{\sqrt{15}} \approx 0.9992$. (That's probably not a great estimate because of my rounding.)

12. Our DE is $0.2Q''+300Q'+100000Q=0$ with $Q(0)=10^{-6}$ and $Q'(0)=0$. We get $0.2r^2+300r+100000=0$, so $r=\dfrac{-300\pm\sqrt{90000-4(.2)(100000)}}{0.4}=\dfrac{-300\pm 100}{0.4}$, so $r=-1000$ or $r=-500$. The general solution is $Q(t)=c_1e^{-1000t}+c_2e^{-500t}$. $Q(0)=10^{-6}=c_1+c_2$, and $Q'(0)=0=-1000c_1-500c_2$. Thus $c_2=-2c_1$ and $c_1=-10^{-6}$. We have

$$Q(t)=10^{-6}(2e^{-500t}-e^{-1000t}).$$

16. $r\sin(\omega_0 t-\theta)=r(\sin\omega_0 t\cos\theta-\sin\theta\cos\omega_0 t)$, so $r\cos\theta=B$ and $-r\sin\theta=A$. Therefore, $r=\sqrt{A^2+B^2}=R$ and $\tan\theta=-\dfrac{A}{B}=-\dfrac{1}{\tan\delta}$. Thus $\theta=\delta+\dfrac{\pi}{2}+2n\pi$ for some integer n.

24. First rewrite the equation as $u''+\dfrac{2k}{3}u=0$. The period is $\pi=\dfrac{2\pi}{\sqrt{\frac{2k}{3}}}$, so $k=6$. We have $u''+4u=0$, so $u(t)=c_1\cos 2t+c_2\sin 2t$. $u(0)=2$, so $c_1=2$. $u'(0)=v$, so $2c_2=v$. Thus $u(t)=2\cos 2t+\dfrac{v}{2}\sin 2t$. The amplitude of this motion is $\sqrt{2^2+\left(\frac{v}{2}\right)^2}=3$, so $v=\pm 2\sqrt{5}$.