Solutions to Homework Assignment 15

MATH 256-01
Section 4.1, Page 212
Problems: 7-10, 11, 13, 15, 21-24, 27, 28

7. The Wronskian is

$$W = \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix} = (2t-3)(8t-8t+2) - 2(4t^2+4-4t^2+2t) = -14.$$

Since this is never 0, the functions are linearly independent.

8. We have

$$W = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix} = (2t-3)(24t-24t-4) - 2(12t^2+6-12t^2-4t) = 0,$$

so the functions are linearly dependent.

9. We have

$$W = \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t & t^2+t+1 \\ 2 & 2t & 4t-1 & 2t+1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0,$$

so the functions are linearly dependent.

10. We have

$$W = \begin{vmatrix} 2t-3 & t^3+1 & 2t^2-t & t^2+t+1 \\ 2 & 3t^2 & 4t-1 & 2t+1 \\ 0 & 6t & 4 & 2 \\ 0 & 6 & 0 & 0 \end{vmatrix} = 156$$

according to MAPLE. Thus, these functions are linearly independent.

11. $y = 1$ is certainly a solution. Also, $(\sin t)''' = -cost$ and $(\sin t)' = \cos t$, so $y = \sin t$ is a solution. $(\cos t)' = -\sin t$ and $(\cos t)''' = \sin t$, so $y = \cos t$ is a solution. The Wronskian is

$$\begin{vmatrix} 1 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = -\cos^2 t - \sin^2 t = -1.$$

13. $(e^t)^{(n)} = e^t$, so it is easy to check that $y = e^t$ is a solution. Likewise, $(e^{-t})^{(n)} = (-1)^n e^{-t}$, so $y = e^{-t}$ also checks. Finally, $(e^{-2t})^{(n)} = (-2)^n e^{2t}$, so this will check out as well. The Wronskian is

$$W = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = -6e^{-2t}.$$

15. $y = 1$ and $y = x$ are clearly solutions since in both cases $y'' = y''' = 0$. If $y = x^3$, then $y''' = 6$ and $y'' = 6x$, so this is also a solution. The Wronskian is

$$W = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x.$$

21. The Wronskian would be $W = ce^{-\int 2dt} = ce^{-2t}$.

22. $W = ce^{-\int 0dt} = c.$

23. $W = ce^{-\int (2/t)dt} = ce^{-2\ln|t|} = \dfrac{c}{t^2}.$

24. $W = ce^{-\int (1/t)dt} = ce^{-\ln|t|} = \dfrac{c}{|t|}.$

27. Let $y = ve^t$. Then $y' = v'e^t + ve^t, y'' = v''e^t + 2v'e^t + ve^t$, and $y''' = v'''e^t + 3v''e^t + 3v'e^t + ve^t$. We get

$$\begin{aligned}&(2-t)e^t(v''' + 3v'' + 3v' + v) + (2t-3)e^t(v'' + 2v' + v) - te^t(v' + v) + ve^t \\ &= e^t[(2-t)v''' + (3-t)v''] \\ &= 0.\end{aligned}$$

Let $u = v''$, so $u' = v'''$. We have $\dfrac{u'}{u} = -\dfrac{t-3}{t-2} = -1 + \dfrac{1}{t-2}$, so $\ln|u| = -t + \ln|t-2| + c$, and $u = \pm c(t-2)e^{-t}$; we may take $u = te^{-t}$, so $v'' = te^{-t}$. Now $v' = -te^{-t} + e^{-t} + c_1$ and $v = te^{-t} + c_1 t + c$; we may take $y_2 = te^{-t}$ and $y_3 = t$.

28. Let $y = vt^2$. Then $y' = v't^2 + 2tv, y'' = v''t^2 + 4tv' + 2v, y''' = v'''t^2 + 6tv'' + 6v'$. We get

$$\begin{aligned}&t^2(t+3)(v'''t^2 + 6tv'' + 6v') - 3t(t+2)(v''t^2 + 4tv' + 2v) + 6(1+t)(v't^2 + 2tv) - 6vt^2 \\ &= t^4(t+3)v''' + (6t^3(t+3) - 3t^3(t+2))v'' + (6t^2(t+3) - 12t^2(t+2) + 6t^2(t+1))v' \\ &= t^4(t+3)v''' + 3t^3(t+4)v'' \\ &= 0.\end{aligned}$$

We now have the first-order equation $t(t+3)v''' + 3(t+4)v'' = 0$ in v''.
We get $\dfrac{v'''}{v''} = -\dfrac{3(t+4)}{t(t+3)} = -\dfrac{4}{t} + \dfrac{1}{t+3}$. Thus $\ln(v'') = -4\ln(t) + \ln(t+3) + c$ and $v'' = c_1\dfrac{t+3}{t^4} = \dfrac{c_1}{t^3} + \dfrac{3c_1}{t^4}$. Thus $v' = -\dfrac{c_1}{2t^2} - \dfrac{c_1}{t^3} + c_2$ and $v = \dfrac{c_1}{2t} + \dfrac{c_1}{2t^2} + c_2 t + c_3$. Multiplying by t^2 gives $\dfrac{c_1}{2}(t+1) + c_2 t^3 + c_3 t^2$. Since we already have t^2 and t^3, our new solution is $t+1$.