Solutions to Homework Assignment 20

MATH 256-01
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Problems: 1-11 odd, 15

1. First, note that the right-hand side is not continuous on $(0, \pi)$, so I will assume that they intended $(0, \pi/2)$. The corresponding homogeneous equation has solutions $1, \cos t$, and $\sin t$. The Wronskian of these is 1. We also get

$$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1,$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t,$$

and

$$W_3 = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -sint.$$

We get

$$\begin{aligned} Y(t) &= 1\int 1 \cdot \tan t dt + \cos t \int (-\cos t)(\tan t)dt + \sin t \int (-\sin t)(\tan t)dt \\ &= -\ln\cos t + \cos^2 t - \sin t \int \frac{1-\cos^2 t}{\cos t}dt \\ &= -\ln\cos t + \cos^2 t - \sin t(\ln(\sec t + \tan t) - \sin t) \\ &= -\ln\cos t - \sin(t)\ln(\sec t + \tan t) + 1. \end{aligned}$$

Thus, the general solution is $y(t) = c_1 + c_2\cos t + c_3 \sin t - \ln\cos t - \sin(t)\ln(\sec t + \tan t)$.

3. $y_1 = e^t, y_2 = e^{-t}$, and $y_3 = e^{2t}$ solve the corresponding homogeneous equation. Their Wronskian is $-6e^{2t}$. Also,

$$W_1 = \begin{vmatrix} 0 & e^{-t} & e^{2t} \\ 0 & -e^{-t} & 2e^{2t} \\ 1 & e^{-t} & 4e^{2t} \end{vmatrix} = 3e^t,$$

$$W_2 = \begin{vmatrix} e^t & 0 & e^{2t} \\ e^t & 0 & 2e^{2t} \\ e^t & 1 & 4e^{2t} \end{vmatrix} = -e^{3t},$$

and

$$W_3 = \begin{vmatrix} e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \\ 1 & e^{-t} & 1 \end{vmatrix} = -2.$$

We get

$$\begin{aligned} Y(t) &= e^t \int \frac{e^{4t}(3e^t)}{-6e^{2t}}dt + e^{-t}\int \frac{e^{4t}(-e^{3t})}{-6e^{2t}}dt + e^{2t}\int \frac{e^{4t}(-2)}{-6e^{2t}}dt \\ &= -\frac{1}{2}e^t(e^{3t}/3) + \frac{1}{6}e^{-t}(e^{5t}/5) + \frac{1}{3}e^{2t}(e^{2t}/2) \\ &= \frac{1}{30}e^{4t}. \end{aligned}$$

The general solution is $y(t) = c_1e^t + c_2e^{-t} + c_3e^{2t} + \dfrac{1}{30}e^{4t}$.

5. $y_1 = e^t, y_2 = \cos t$, and $y_3 = \sin t$ solve the corresponding homogeneous equation. We have $W = 2e^t, W_1 = 1, W_2 = -e^t(\cos t - \sin t)$, and $W_3 = -e^t(\sin t + \cos t)$. We get

$$\begin{aligned} Y(t) &= e^t \int \frac{e^{-t}\sin t(1)}{2e^t} dt + \cos t \int \frac{e^{-t}\sin t(-e^t(\cos t - \sin t))}{2e^t} dt + \sin t \int \frac{e^{-t}\sin t(-e^t(\sin t + \cos t))}{2e^t} dt \\ &= \frac{1}{2}e^t \int e^{-2t}\sin t dt - \frac{1}{2}\cos t \int e^{-t}(\sin t\cos t - \sin^2 t)dt - \frac{1}{2}\sin t \int e^{-t}(\sin^2 t + \sin t\cos t)dt \\ &= -\frac{1}{5}e^{-t}\cos t \end{aligned}$$

after a bout with MAPLE. (Use some trig identities and integrate by parts twice, if you dare.)

The general solution is $y(t) = c_1 e^t + c_2\cos t + c_3 \sin t - \frac{1}{5}e^{-t}\cos t$.

7. Solutions to the corresponding homogeneous equation are the same as in Number 5. The general solution is therefore

$$\begin{aligned} y(t) &= c_1e^t + c_2\cos t + c_3\sin t + e^t \int \frac{\sec t}{2e^t} dt + \cos t \int \frac{\sec t(-e^t(\cos t - \sin t))}{2e^t} dt + \sin t \int \frac{\sec t(-e^t(\sin t + \cos t))}{2e^t} dt \\ &= c_1e^t + c_2\cos t + c_3\sin t + \frac{1}{2}e^t \int e^{-t}\sec t dt + \frac{1}{2}\cos t(-t - \ln(\cos t)) + \frac{1}{2}\sin t(\ln\cos t - t). \end{aligned}$$

9. I need to go! Use the solution from number 4 and apply the initial conditions.

11. Use the solution from number 6.

15. MAPLE tells me that if $y_1 = \sin t, y_2 = \cos t, y_3 = \sinh t$, and $y_4 = \cosh t$, then $W = 4, W_1 = -2\cos t, W_2 = 2\sin t, W_3 = 2\cosh t$, and $W_4 = -2\sinh t$. Thus
$Y(t) = \sin t \int \frac{-2g(t)\cos t}{4} dt + \cos t \int \frac{2g(t)\sin t}{4} dt + \sinh t \int \frac{2g(t)\cosh t}{4} dt + \cosh t \int \frac{-2g(t)\sinh t}{4} dt.$
This can be simplified somewhat with identities.