Solutions to Homework Assignment 22

MATH 256-01
Section 5.1, Page 237
Problems: 1-16 odd, 19-28

1. The ratio of consecutive coefficients is 1, so the radius of convergence is $1/1 = 1$.

2. We need $\lim_{n\to\infty} \dfrac{(n+1)/2^{n+1}}{n/2^n} = \dfrac{1}{2}$, so the radius of convergence is 2.

3. We have $\lim_{n\to\infty} \dfrac{1/(n+1)!}{/}(1/n! = \lim_{n\to\infty} \dfrac{1}{n+1} = 0$, so $\rho = \infty$.

4. $\lim_{n\to\infty} \dfrac{2^{n+1}}{2^n} = 2$, so $\rho = \dfrac{1}{2}$.

5. The nth term is $\dfrac{2^n(x+1/2)^n}{n^2}$. We have $\lim_{n\to\infty} \dfrac{2^{n+1}n^2}{2^n n^2} = 2$, so $\rho = \dfrac{1}{2}$.

6. $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$, so $\rho = 1$.

7. $\lim_{n\to\infty} \dfrac{(n+1)^2 3^n}{n^2 3^{n+1}} = \dfrac{1}{3}$, so $\rho = 3$.

8. $$\begin{aligned}\lim_{n\to\infty} \frac{(n+1)!n^n}{n!(n+1)^{n+1}} &= \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\ &= \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^n\right]^{-1} \\ &= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} \\ &= e^{-1}.\end{aligned}$$

   Thus, $\rho = e$.

9. We have $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f^{(4)}(x) = \sin x$, after which the sequence repeats. Thus $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$. Thus $a_n = \dfrac{(-1)^n}{n!}$ if n is odd and 0 if n is even. We get $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{(2n+1)}}{(2n+1)!}$. The radius of convergence is infinite.

10. We did e^x in class; all derivatives are just e^x (giving 1 at 0), so $a_n = \dfrac{1}{n!}$. Thus $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. The radius of convergence is infinite.

11. $f(x) = x, f'(x) = 1$, and $f^{(n)}(x) = 0$ for all $n > 1$. Thus $a_0 = 1, a_1 = 1$, and $a_n = 0$ for $n > 1$. Therefore, $x = 1 + 1(x-1)^1 = 1 + (x-1)$. The radius of convergence is infinite.

12. $f(x) = x^2, f'(x) = 2x, f''(x) = 2$, and $f^{(n)}(x) = 0$ for $n > 2$. Thus $a_0 = 1, a_1 = -2, a_2 = 2/2! = 1$, and $a_n = 0$ for $n > 2$. We have $x^2 = 1 - 2(x+1) + 1(x+1)^2$. The radius of convergence is again infinite.

13. $f(x) = \ln(x), f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}$, and, in general, $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$. Evaluating at $x = 1$ gives $a_n = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$. The series is therefore $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$. The radius of convergence is 1.

14. We saw in class that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$. The radius of convergence is 1.

15. We did this in class.

16. We may write $\frac{1}{1-x} = \frac{-1}{1-(-(x-2))} = -\sum_{n=0}^{\infty}(-1)^n(x-2)^n$.

19. If we begin summing 1 higher, we need to decrease the index of summation by 1 to compensate, so these match.

20.

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_1 + \sum_{k=1}^{\infty} a_{k+1}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_1 + \sum_{k=1}^{\infty}(a_{k+1} + a_{k-1})x^k.\end{aligned}$$

21. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)x^n$

22. $\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2}x^n$.

23. $x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} na_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty}(n+1)a_n x^n = \sum_{n=1}^{\infty}(n+1)a_n x^n$.

24. We get

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n.\end{aligned}$$

25. We get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + na_n]x^n.$$

26. We have

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^{n+1} &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= a_1 + \sum_{n=0}^{\infty}[(n+1)a_{n+1} + a_{n-1}]x^n.\end{aligned}$$

27. We have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=0}^{\infty}[n(n+1)a_{n+1} + a_n]x^n.$$

28. We have

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = \qquad = \sum_{n=0}^{\infty}[(n+1)a_{n+1} + 2a_n]x^n$$
$$= 0.$$

We require that each coefficient be zero; thus, $a_{n+1} = -\frac{2a_n}{n+1}$. This means $a_0 = a_0, a_1 = -\frac{2a_0}{1}, a_2 = \frac{4a_0}{2}, a_3 = -\frac{8a_0}{6}, \ldots$. In general, $a_n = (-1)^n \frac{2^n a_0}{n!}$. The function is therefore $a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = a_0 e^{-2x}$.