Solutions to Homework Assignment 23

MATH 256-01
Section 5.2, Page 247
Problems: 2-7, 9-14, 16, 21

2. $y'' - xy' - y = 0$ about $x_0 = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$\begin{aligned}
y'' - xy' - y &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n \\
&= (a_2 \cdot 2 \cdot 1 - a_0) + \sum_{n=1}^{\infty} [a_{n+2}(n+2)(n+1) - a_n n - a_n]x^n \\
&= (a_2 \cdot 2 \cdot 1 - a_0) + \sum_{n=1}^{\infty} [a_{n+2}(n+2)(n+1) - a_n(n+1)]x^n \\
&= 0.
\end{aligned}$$

Thus $a_2 = \frac{1}{2}a_0$ and for $n \geq 1$, $a_{n+2} = \frac{a_n}{n+2}$. This gives $a_0 = a_0, a_1 = a_1, a_2 = \frac{1}{2}a_0, a_3 = \frac{1}{3}a_1, a_4 = \frac{1}{4}a_2 = \frac{1}{8}a_0, a_5 = \frac{1}{5}a_3 = \frac{1}{15}a_1, a_6 = \frac{1}{6}a_4 = \frac{1}{48}a_0, \ldots$.

We have $y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2 \cdot 4 \cdot 6 \cdots 2n} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$.

This gives $y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \ldots$ and $y_2 = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \ldots$.

3. This time, $y = \sum_{n=0}^{\infty} a_n(x-1)^n$. Thus $y' = \sum_{n=1}^{\infty} a_n n(x-1)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}$. We get

$$\begin{aligned}
y'' - xy' - y &= \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - x\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - (1 + (x-1))\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=1}^{\infty} a_n n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n - \sum_{n=1}^{\infty} a_n n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= (a_2 \cdot 2 \cdot 1 - a_1 \cdot 1 - a_0) + \sum_{n=1}^{\infty} (a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n n - a_n)x^n \\
&= (2a_2 - a_1 - a_0) + \sum_{n=1}^{\infty} (a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n(n+1))x^n.
\end{aligned}$$

Thus $a_2 = \frac{a_0 + a_1}{2}$ and $a_{n+2} = \frac{a_{n+1} + a_n}{n+2}$ for $n \geq 1$. We get $a_0 = a_0, a_1 = a_1, a_2 = \frac{a_1 + a_0}{2}, a_3 = \frac{a_2 + a_1}{3} = \frac{\frac{a_1+a_0}{2} + a_1}{3} = \frac{3a_1 + a_0}{6}, a_4 = \frac{a_3 + a_2}{4} = \frac{\frac{3a_1+a_0}{6} + \frac{a_1+a_0}{2}}{4} = \frac{6a_1 + 4a_0}{24}, \ldots$. Thus $y(t) = a_0\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 \ldots\right) + a_1\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \ldots\right)$.

These give us $y_1(t) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 \ldots$ and $y_2(t) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \ldots$.

4. $y'' + k^2x^2y = 0$ about $x_0 = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$\begin{aligned}
y'' + k^2x^2y &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + k^2x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n k^2 x^{n+2} \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} a_{n-2} k^2 x^n \\
&= (a_2 \cdot 2 \cdot 1) + (a_3 \cdot 3 \cdot 2x) + \sum_{n=2}^{\infty}(a_{n+2}(n+2)(n+1) + a_{n-2}k^2)x^n.
\end{aligned}$$

Thus $a_0 = a_0, a_1 = a_1, a_2 = 0, a_3 = 0$, and $a_{n+2} = -\frac{a_{n-2}k^2}{(n+2)(n+1)}$ for $n \geq 2$. This gives $a_4 = -\frac{a_0k^2}{4 \cdot 3}, a_5 = -\frac{a_1k^2}{5 \cdot 4}, a_6 = 0, a_7 = 0, a_8 = -\frac{a_4k^2}{8 \cdot 7} = \frac{a_0k^4}{8 \cdot 7 \cdot 4 \cdot 3}, a_9 = -\frac{a_5k^2}{9 \cdot 8} = \frac{a_1k^4}{9 \cdot 8 \cdot 5 \cdot 4}, \ldots$. We now have

$$\begin{aligned}
y(t) = a_0 &\left(1 - \frac{k^2}{4 \cdot 3}x^4 + \frac{a_0k^4}{8 \cdot 7 \cdot 4 \cdot 3}x^8 - \frac{a_0k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}x^{12} + \ldots\right) \\
+a_1 &\left(x - \frac{k^2}{5 \cdot 4}x^5 + \frac{a_1k^4}{9 \cdot 8 \cdot 5 \cdot 4}x^9 - \frac{a_1k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}x^{13} + \ldots\right).
\end{aligned}$$

5. $(1-x)y'' + y = 0$ about $x_0 = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$\begin{aligned}
(1-x)y'' + y &= (1-x)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n+1}(n+1)n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= (a_2 \cdot 2 \cdot 1 + a_0) + \sum_{n=1}^{\infty}[a_{n+2}(n+2)(n+1) - a_{n+1}(n+1)n + a_n]x^n.
\end{aligned}$$

Thus $a_2 = -\frac{1}{2}a_0$ and for $x \geq 1$, $x_{n+2} = \frac{a_{n+1}(n+1)n - a_n}{(n+2)(n+1)}$. This gives $a_0 = a_0, a_1 = a_1, a_2 =$

$-\dfrac{1}{2}a_0, a_3 = \dfrac{a_2(2)(1) - a_1}{3 \cdot 2} = \dfrac{-a_0 - a_1}{3!}, a_4 = \dfrac{a_3(3)(2) - a_2}{4 \cdot 3} = \dfrac{(-a_0 - a_1) + \frac{1}{2}a_0}{4 \cdot 3} = \dfrac{-a_0 - 2a_1}{4!}, a_5 = \dfrac{a_4(4)(3) - a_3}{5 \cdot 4} = \dfrac{-\frac{1}{2}a_0 - a_1 - \frac{1}{6}(-a_0 - a_1)}{5 \cdot 4} = \dfrac{-2a_0 - 5a_1}{5!}, \ldots$. Therefore
$y(t) = a_0\left(1 - \dfrac{1}{2!}x^2 - \dfrac{1}{3!}x^3 - \dfrac{1}{4!}x^4 + \ldots\right) + a_1\left((x - \dfrac{1}{3!}x^3 - \dfrac{2}{4!}x^4 - \dfrac{1}{24}x^5 + \ldots\right)$.

6. I did this one in class.

7. $y'' + xy' + 2y = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$\begin{aligned}
y'' + xy' + 2y &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\
&= (a_2 \cdot 2 \cdot 1 + 2a_0) + \sum_{n=1}^{\infty}(a_{n+2}(n+2)(n+1) + a_n n + 2a_n)x^n \\
&= 0.
\end{aligned}$$

Thus $a_2 = -a_0$ and for $n \geq 1$, $a_{n+2} = -\dfrac{a_n}{n+1}$. Now $a_0 = a_0, a_1 = a_1, a_2 = -a_0, a_3 = -\dfrac{a_1}{2}, a_4 = -\dfrac{a_2}{3} = \dfrac{a_0}{3}, a_5 = -\dfrac{a_3}{4} = \dfrac{a_1}{2 \cdot 4}, a_6 = -\dfrac{a_4}{5} = -\dfrac{a_0}{3 \cdot 5}, a_7 = -\dfrac{a_5}{6} = -\dfrac{a_1}{2 \cdot 4 \cdot 6}, \ldots$. In general, we have $a_{2n+1} = (-1)^n \dfrac{a_1}{2^n n!}$ and $a_{2n} = (-1)^n \dfrac{2^n n! a_0}{(2n)!}$.

Finally, we get $y(t) = a_0 \sum_{n=0}^{\infty} (-1)^n \dfrac{2^n n!}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2^n n!} x^{2n+1}$.

9. $(1 + x^2)y'' - 4xy' + 6y = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.
We get

$$\begin{aligned}
(1 + x^2)y'' - 4xy' + 6y &= (1 + x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 4x\sum_{n=1}^{\infty} a_n n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=1}^{\infty} 4a_n n x^n + \sum_{n=0}^{\infty} 6a_n x^n \\
&= (a_2 \cdot 2 \cdot 1 + 6a_0) + (a_3 \cdot 3 \cdot 2 + 4a_1 \cdot 1 + 6a_1)x \\
&\quad + \sum_{n=2}^{\infty}(a_{n+2}(n+2)(n+1) + a_n n(n-1) - 4a_n n + 6a_n)x^n \\
&= (2a_2 + 6a_0) + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty}[a_{n+2}(n+2)(n+1) + a_n(n^2 - 5n + 6)]x^n \\
&= 0.
\end{aligned}$$

We get $a_2 = -3a_0, a_3 = -\dfrac{1}{3}a_1$, and $a_{n+2} = -\dfrac{a_n(n-2)(n-3)}{(n+2)(n+1)}$. Thus $a_0 = a_0, a_1 = a_1, a_2 = -3a_0, a_3 = -\dfrac{1}{3}a_1, a_4 = 0, a_5 = 0, \ldots$. Thus $y(t) = a_0(1 - 3x^2) + a_1\left(x - \dfrac{1}{3}x^3\right)$.

10. $(4-x^2)y''+2y=0$. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Then $y'=\sum_{n=1}^{\infty}a_nnx^{n-1}$ and $y''=\sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}$. We get

$$\begin{aligned}(4-x^2)y''+2y &= (4-x^2)\sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}+2\sum_{n=0}^{\infty}a_nx^n \\ &= \sum_{n=0}^{\infty}4a_{n+2}(n+2)(n+1)x^n-\sum_{n=2}^{\infty}a_nn(n-1)x^n+\sum_{n=0}^{\infty}2a_nx^n \\ &= (4a_2\cdot 2\cdot 1+2a_0)+(a_3\cdot 3\cdot 2+2a_1)x+\sum_{n=2}^{\infty}(a_{n+2}(n+2)(n+1)-a_nn(n-1)+2a_n)x^n \\ &= (4a_2\cdot 2\cdot 1+2a_0)+(4a_3\cdot 3\cdot 2+2a_1)x+\sum_{n=2}^{\infty}[4a_{n+2}(n+2)(n+1)-a_n(n^2-n+2)]x^n \\ &= 0.\end{aligned}$$

We get $a_2=-\frac{1}{4}a_0, a_3=-\frac{1}{4\cdot 3}a_1$, and for $n\geq 2$, $a_{n+2}=\frac{a_n(n-2)(n+1)}{4(n+2)(n+1)}$. This gives $a_4=0, a_5=\frac{4a_3}{4\cdot 5\cdot 4}=-\frac{a_1}{4^2\cdot 5\cdot 3}, a_6=0, a_7=\frac{a_5\cdot 3\cdot 6}{4\cdot 7\cdot 6}=-\frac{a_1}{4^3\cdot 7\cdot 5}, \ldots$. In general, $a_{2n}=0$ for $n\geq 2$ and $a_{2n+1}=-\frac{a_1}{4^n(2n+1)(2n-1)}$. Thus $y(t)=a_0\left(1-\frac{1}{4}x^2\right)+a_1\sum_{n=0}^{\infty}\frac{-1}{4^n(2n+1)(2n-1)}x^{2n+1}$.

11. $(3-x^2)y''-3xy'-y=0$. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Then $y'=\sum_{n=1}^{\infty}a_nnx^{n-1}$ and $y''=\sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}$. We get

$$\begin{aligned}(3-x^2)y''-3xy'-y &= \sum_{n=0}^{\infty}3a_{n+2}(n+2)(n+1)x^n-\sum_{n=2}^{\infty}a_nn(n-1)x^n-\sum_{n=1}^{\infty}3a_nnx^n-\sum_{n=0}^{\infty}a_nx^n \\ &= (6a_2-a_0)+(18a_3-3a_1-a_1)x+\sum_{n=2}^{\infty}[3a_{n+2}(n+2)(n+1)-a_nn(n-1)-3a_nn-a_n]x^n \\ &= (6a_2-a_0)+(18a_3-4a_1)x+\sum_{n=2}^{\infty}[3a_{n+2}(n+2)(n+1)-a_n(n^2-n+3n+1]x^n \\ &= 0.\end{aligned}$$

We get $a_2=\frac{1}{3\cdot 2}a_0, a_3=\frac{2}{3\cdot 3}a_1$, and $a_{n+2}=\frac{a_n(n+1)}{3(n+2)}$. Thus $a_4=\frac{a_2}{4}=\frac{a_0}{3\cdot 2\cdot 4}, a_5=\frac{a_3\cdot 4}{3\cdot 5}=\frac{2\cdot 4a_1}{3^2\cdot 3\cdot 5}, a_6=\frac{a_4\cdot 5}{3\cdot 6}=\frac{a_0\cdot 5}{3^2\cdot 2\cdot 4\cdot 6}, a_7=\frac{a_5\cdot 6}{3\cdot 7}=\frac{2\cdot 4\cdot 6a_1}{3^3\cdot 3\cdot 5\cdot 7}, \ldots$. Whew!
Thus $y(x)=a_0\sum_{n=0}^{\infty}\frac{3\cdot 5\cdots(2n-1)}{3^n\cdot 2\cdot 4\cdots(2n)}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{2\cdot 4\cdots(2n)}{3^n\cdot 3\cdot 5\cdots(2n+1)}x^{2n+1}$.

12. $(1-x)y''+xy'-y=0$. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Then $y'=\sum_{n=1}^{\infty}a_nnx^{n-1}$ and $y''=\sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}$. We get

$$
\begin{aligned}
&(1-x)y'' + xy' - y \\
&= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n+1} n(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n \\
&= (2a_2 - a_0) + \sum_{n=1}^{\infty} [a_{n+2}(n+2)(n+1) - a_{n+1} n(n+1) + a_n n - a_n] x^n \\
&= 0.
\end{aligned}
$$

Thus $a_2 = \frac{1}{2}a_0$ and for $n \geq 1$, $a_{n+2} = \frac{a_{n+1}n(n+1) - a_n(n-1)}{(n+2)(n+1)}$.

We get $a_0 = a_0, a_1 = a_1, a_2 = \frac{1}{2}a_0, a_3 = \frac{2a_2 - 0}{6} = \frac{1}{3!}a_0, a_4 = \frac{6a_3 - a_2}{4 \cdot 3} = \frac{1}{4!}a_0, a_5 = \frac{12a_4 - 2a_3}{5 \cdot 4} = \frac{1}{5!}a_0, \ldots$. We get $y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots = e^x - x$, and $y_2 = x$. (Notice that a_1 only appears once!)

13. $2y'' + xy' + 3y = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$
\begin{aligned}
&2y'' + xy' + 3y \\
&= 2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + 3\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} 2a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 3a_n x^n \\
&= (4a_2 + 3a_0) + \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + a_n(n+3)] x^n \\
&= 0.
\end{aligned}
$$

Thus $a_2 = -\frac{3}{4}a_0$ and for $n \geq 1$, $a_{n+2} = -\frac{a_n(n+3)}{2(n+2)(n+1)}$.

We get $a_0 = a_0, a_1 = a_1, a_2 = -\frac{3}{4}a_0, a_3 = -\frac{4}{2 \cdot \cdot 3!}a_1, a_4 = \frac{5a_2}{2 \cdot 4 \cdot 3} = \frac{3 \cdot 5}{2^2 \cdot 4!}a_0, a_5 = -\frac{6a_3}{2 \cdot 5 \cdot 4} = \frac{4 \cdot 6}{2^3 \cdot 5!}a_1, a_6 = -\frac{7a_4}{2 \cdot 6 \cdot 5} = \frac{3 \cdot 5 \cdot 7}{2^3 \cdot 6!}a_0, \ldots$. Thus

$$y_1(x) = 1 - \frac{3}{4}x^2 - \frac{3 \cdot 5}{2^2 \cdot 4!}x^4 + \ldots + (-1)^n \frac{3 \cdot 5 \cdots (2n+1)}{2^n (2n)!} x^{2n} + \ldots$$

and

$$y_2(x) = x - \frac{4}{2 \cdot 3!}x^3 + \frac{4 \cdot 6}{2^3 \cdot 5!}x^5 + \ldots + (-1)^n \frac{4 \cdot 6 \cdots 2n}{(2n+1)!} x^{2n+1} + \ldots.$$

14. $2y'' + (x+1)y' + 3y = 0, x_0 = 2$. Let $y = \sum_{n=0}^{\infty} a_n (x-2)^n$. Then $y' = \sum_{n=1}^{\infty} a_n n (x-2)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x-2)^{n-2}$. We get

$$\begin{aligned}
&2y'' + (x+1)y' + 3y \\
&= 2\sum_{n=2}^{\infty} a_n n(n-1)(x-2)^{n-2} + (x-2+3)\sum_{n=1}^{\infty} a_n n(x-2)^{n-1} + 3\sum_{n=0}^{\infty} a_n (x-2)^n \\
&= \sum_{n=0}^{\infty} 2a_{n+2}(n+2)(n+1)(x-2)^n + \sum_{n=1}^{\infty} a_n n(x-2)^n + \sum_{n=0}^{\infty} 3a_{n+1}(n+1)(x-2)^n + \sum_{n=0}^{\infty} 3a_n(x-2)^n \\
&= (4a_2 + 3a_1 + 3a_0) + \sum_{n=1}^{\infty}[2a_{n+2}(n+2)(n+1) + a_n n + 3a_{n+1}(n+1) + 3a_n](x-2)^n \\
&= 0.
\end{aligned}$$

We get $a_2 = -\dfrac{3a_1 + 3a_0}{4}$ and for $n \geq 1$, $a_{n+2} = -\dfrac{3a_{n+1}(n+1) + a_n(n+3)}{2(n+2)(n+1)}$. Thus $a_0 = a_0, a_1 = a_1, a_2 = -\dfrac{3a_1 + 3a_0}{4}, a_3 = -\dfrac{6a_2 + 4a_1}{12} = -\dfrac{6\frac{3a_1+3a_0}{4} + 4a_1}{12} = -\dfrac{-\frac{1}{2}a_1 - \frac{9}{2}a_0}{12} = \dfrac{a_1}{24} + \dfrac{3}{8}a_0$, $a_4 = -\dfrac{9a_3 + 5a_2}{24} = -\dfrac{3}{8}\left(\dfrac{a_1}{24} + \dfrac{3}{8}a_0\right) - \dfrac{5}{24}\left(-\dfrac{3a_1 + 3a_0}{4}\right) = \dfrac{9}{64}a_1 + \dfrac{1}{64}a_0, \ldots.$

Thus $y_1(x) = 1 - \dfrac{3}{4}(x-2)^2 + \dfrac{3}{8}(x-2)^3 + \dfrac{1}{64}(x-2)^4 + \ldots$ and $y_2(x) = (x-2) - \dfrac{3}{4}(x-2)^2 + \dfrac{1}{24}(x-2)^3 + \dfrac{9}{64}(x-2)^4 + \ldots.$

16. I did Number 6 in class; you can check the initial values yourself.

21. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

 (a)

$$\begin{aligned}
&y'' - 2xy' + \lambda y \\
&= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n \\
&= (2a_2 + \lambda a_0) + \sum_{n=1}^{\infty}(a_{n+2}(n+2)(n+1) - 2a_n n + \lambda a_n)x^n \\
&= 0.
\end{aligned}$$

 Thus $a_2 = -\dfrac{1\lambda}{2}a_0$ and for $n \geq 1$, $a_{n+2} = \dfrac{(2n-\lambda)a_n}{(n+2)(n+1)}$.

 We get $a_0 = a_0, a_1 = a_1, a_2 = -\dfrac{\lambda}{2}a_0, a_3 = \dfrac{(2-\lambda)a_1}{3!}, a_4 = \dfrac{\lambda(4-\lambda)a_2}{12} = -\dfrac{\lambda(4-\lambda)a_0}{4!}, a_5 = \dfrac{(6-\lambda)a_3}{20} = \dfrac{(2-\lambda)(6-\lambda)a_1}{5!}, a_6 = \dfrac{(8-\lambda)a_4}{30} = -\dfrac{\lambda(4-\lambda)(8-\lambda)a_0}{6!}$,

 $a_7 = \dfrac{(10-\lambda)a_5}{42} = \dfrac{(2-\lambda)(6-\lambda)(10-\lambda)a_1}{7!}, \ldots.$

 Thus $y_1(x) = 1 - \dfrac{\lambda}{2}x^2 - \dfrac{\lambda(4-\lambda)}{24}x^4 - \dfrac{\lambda(4-\lambda)(8-\lambda)}{720}x^6 - \ldots$ and $y_2(x) = x + \dfrac{2-\lambda}{6}x^3 + \dfrac{(2-\lambda)(6-\lambda)}{120}x^5 + \dfrac{(2-\lambda)(6-\lambda)(10-\lambda)}{5040}x^7 + \ldots.$

 (b) If λ is an even integer, then $\lambda - 2n = 0$ for some n, and the series will terminate.

  i. $\lambda = 0 : y_1(x) = 1.$

ii. $\lambda = 2 : y_2(x) = x$.

iii. $\lambda = 4 : y_1(x) = 1 - 2x^2$.

iv. $\lambda = 6 : y_2(x) = x - \dfrac{2}{3}x^3$.

v. $\lambda = 8 : y_1(x) = 1 - 4x^2 + \dfrac{4}{3}x^4$.

vi. $\lambda = 10 : y_2(x) = x - \dfrac{4}{3}x^3 + \dfrac{4}{15}x^5$.