Solutions to Homework Assignment 24

MATH 256-01
Section 5.3, Page 253
Problems: 1, 5-10, 12, 13, 15, 16, 18, 21

1. $\phi''(x) = -x\phi'(x) - \phi(x)$, so $\phi''(0) = -\phi(0) = -1$. $\phi'''(x) = -\phi'(x) - x\phi''(x) - \phi'(x)$, so $\phi'''(0) = -2\phi'(0) = 0$. $\phi^{(4)}(x) = -2\phi''(x) - \phi''(x) - x\phi'''(x)$, so $\phi^{(4)}(0) = -3\phi''(0) = 3$.

5. $p(x)$ and $q(x)$ are analytic on $(-\infty, \infty)$, so the radius of convergence of a solution is ∞ for both points.

6. $p(x) = \dfrac{x}{(x-3)(x+1)}$. This is analytic everywhere except $x = 3, -1$. There is a series solution at $x_0 = 4$ with a radius of convergence of at least $|4-3| = 1$. The solution at $x_0 = -4$ has a radius of convergence of at least $|-4-(-1)| = 3$. The solution at $x_0 = 0$ has a radius of convergence of at least $|0-(-1)| = 1$.

7. The denominator for both p and q is $x^3 + 1 = (x+1)(x^2 - x + 1)$. The zeros of this are $x = -1$ and $x = \dfrac{1 \pm \sqrt{-3}}{2} = \dfrac{1}{2} \pm \sqrt{3}2i$. These are all a distance of 1 from $x_0 = 0$, so the solution at $x_0 = 0$ has a radius of convergence of at least $|0-(-1)| = 1$.

   The distance from $x_0 = 2$ to -1 is $|2-(-1)| = 3$. The distance from $x_0 = 2$ to either of $\dfrac{1}{2} \pm \sqrt{3}2i$ is $\sqrt{\left(\dfrac{1}{2} - 2\right)^2 + \left(\dfrac{\sqrt{3}}{2} - 0\right)^2} = \sqrt{\dfrac{9}{4} + \dfrac{3}{4}} = \sqrt{3}$, which is smaller than 3. Thus, the series at $x_0 = 2$ has a radius of convergence of at least $\sqrt{3}$.

8. $p(x) = 0$ is analytic everywhere, but $q(x) = 0$ is not analytic at $x_0 = 0$. The radius of convergence is $|1-0| = 1$.

9. (a) $p(x)$ and $q(x)$ are constants, so the radius of convergence is ∞ for any choice of x_0.
   (b) Since $P(x)$ is a constant, the radius of convergence is ∞ for any choice of x_0.
   (c) Since $P(x)$ is a constant, the radius of convergence is ∞ for any choice of x_0.
   (d) Since $P(x)$ is a constant, the radius of convergence is ∞ for any choice of x_0.
   (e) The only singular point is $x = 1$, and it is 1 unit from $x_0 = 0$, so $\rho = 1$.
   (f) The singular points are at $x = \pm i\sqrt{2}$. The distance from these to $x_0 = 0$ is $\sqrt{(0-0)^2 + (\sqrt{2}-0)^2} = \sqrt{2}$, so $\rho = \sqrt{2}$.
   (g) Since $P(x)$ is a constant, the radius of convergence is ∞ for any choice of x_0.
   (h) The only singular point is $x_0 = 0$, and it is 1 unit from $x_0 = 1$. Thus $\rho = 1$.
   (i) The singular points are $\pm i$, and they are 1 unit from $x_0 = 0$, so $\rho = 1$.
   (j) The singular points are ± 2, and they are 2 units from $x_0 = 0$, so $\rho = 2$.

(k) The singular points are $\pm\sqrt{3}$, and they are $\sqrt{3}$ units from $x_0 = 0$, so $\rho = \sqrt{3}$.

(l) The only singular point is 1, and it is unit from $x_0 = 0$, so $\rho = 1$.

(m) Since $P(x)$ is a constant, the radius of convergence is ∞ for any choice of x_0.

(n) Since $P(x)$ is a constant, the radius of convergence is ∞ for any choice of x_0.

10. $(1-x^2)y'' - xy' + \alpha^2 y = 0$.

(a) Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$\begin{aligned}(1-x^2)y'' - xy' + \alpha^2 y &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n \qquad = \\ &= (2a_2 + \alpha^2 a_0) + (6a_3 + a_1(\alpha^2 - 1))x + \sum_{n=2}^{\infty} [a_{n+2}(n+2)(n+1) - a_n(n^2 - \alpha^2)]x^n \\ &= 0.\end{aligned}$$

Thus $a_2 = -\dfrac{a_0\alpha^2}{2}$, $a_3 = \dfrac{(1-\alpha^2)a_1}{6}$, and $a_{n+2} = \dfrac{a_n(n-\alpha)(n+\alpha)}{(n+2)(n+1)}$. This gives $a_4 = \dfrac{a_2(4-\alpha^2)}{(4\cdot 3)} = -\dfrac{a_0\alpha^2(2^2-\alpha^2)}{4!}$, $a_5 = \dfrac{a_3(3^2-\alpha^2)}{5\cdot 4} = \dfrac{(1-\alpha^2)(3-\alpha^2)a_1}{5!}, \ldots$. We get the two linearly independent solutions

$$y_1(t) = 1 - \frac{\alpha^2}{2}x^2 - \frac{\alpha^2(2^2-\alpha^2)}{4!}x^4 - \frac{\alpha^2(2^2-\alpha^2)(4^2-\alpha^2)}{6!}x^6 - \ldots$$

and

$$y_2(t) = x - \frac{1-\alpha^2}{3!}x^3 - \frac{(1-\alpha^2)(3^2-\alpha^2)}{5!}x^5 - \ldots.$$

(b) If $\alpha = m$, then one of the series will terminate when $n = m$, leaving just a polynomial. (The other solution remains an infinite series.)

(c) If $\alpha = 0$, then $a_2 = 0$, so $y(x) = 1$ is a polynomial solution. If $\alpha = 1$, then $a_3 = 0$, so $y(x) = x$ is a polynomial solution. If $\alpha = 2$, then $a_4 = 0$, so $y(x) = 1 - 2x^2$ is a solution. If $\alpha = 3$, then $a_5 = 0$, so $y(x) = x - \frac{4}{3}x^3$ is a polynomial solution.

12. The radius of convergence should be infinite since $e^x \neq 0$. $e^x y'' + xy = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$e^x y'' + xy = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}\right) + x\sum_{n=0}^{\infty} a_n x^n$$

$$
\begin{aligned}
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right)\left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots\right) + \bigl(a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \\
&= (2a_2) + (2a_2 + 6a_3 + a_0)x + (12a_4 + 6a_3 + a_2 + a_1)x^2 + \left(20a_5 + 12a_4 + 3a_3 + \frac{1}{3}a_2 + a_2\right)x^3 + \Biggl(30a_6 + 20a_5 + \\
&= 0.
\end{aligned}
$$

Thus $a_0 = a_0, a_1 = a_1, a_2 = 0, a_3 = -\frac{1}{6}a_0, a_4 = -\frac{1}{12}(a_1+6a_3) = -\frac{1}{12}a_1+\frac{1}{12}a_0, a_5 = -\frac{1}{20}\left(-a_1 + a_0 - \frac{1}{2}a_0\right) = \frac{1}{20}a_1 - \frac{1}{40}a_0, a_6 = -\frac{1}{30}\left(a_1 - \frac{1}{2}a_0 - \frac{1}{2}a_1 + \frac{1}{2}a_0 - \frac{1}{6}a_0\right) = -\frac{1}{60}a_1 - \frac{1}{6}a_0, \dots.$

We have $y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + \dots$ and $y_2(x) = 1 - \frac{1}{12}x^4 + \frac{1}{20}x^5 - \frac{1}{60}x^6 + \dots.$

Mercy!

13. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$
\begin{aligned}
&(\cos x)y'' + xy' - 2y \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\left(\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}\right) + x\sum_{n=1}^{\infty} a_n n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \dots\right) + \bigl(a_1x + 2a_2x^2 + 3a_3x^3 + \\
&\quad - \left(2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + 2a_5x^5 \dots\right) \\
&= (2a_2 - 2a_0) + (6a_3 + a_1 - 2a_1)x + (12a_4 - a_2 + 2a_2 - 2a_2)x^2 + (20a_5 - 3a_3 + 3a_3 - 2a_3)x^3 + \Biggl(30a_6 - 6a_4 + \frac{1}{12} \\
&\quad + \left(42a_7 - 10a_5 + \frac{1}{4}a_3 + 5a_5 - 2a_5\right)x^5 + \dots \\
&= 0.
\end{aligned}
$$

$a_0 = a_0, a_1 = a_1, a_2 = a_0, a_3 = \frac{1}{6}a_1, a_4 = \frac{1}{12}a_2 = \frac{1}{12}a_0, a_5 = \frac{a_3}{10} = \frac{1}{60}a_1, a_6 = \frac{1}{30}\left(4a_4 - \frac{1}{12}a_2\right) = \frac{1}{30}\left(\frac{a_0}{3} - \frac{a_0}{12}\right) = \frac{a_0}{120}, a_7 = \frac{1}{42}\left(\frac{7}{60}a_1 - \frac{1}{24}a_1\right) = \frac{1}{560}a_1$. We get

$y_1(x) = 1 + x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \dots$ and $y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{560}x^7 + \dots.$

15. Let $y(x) = c_1x + c_2x^2$. Theorem 3.2.1 guarantees us that we can prescribe the values $y(0)$ and $y'(0)$ to be whatever we want, provided that $p(x)$ and $q(x)$ are continuous at $x = 0$. However, $y(0) = 0$ because of the form of our solution, so we *cannot* choose c_1 and c_2 so that $y(0) = 1$ (for example). Thus, $p(x)$ or $q(x)$ is not continuous at $x = 0$, and therefore also not analytic at $x = 0$, so $x = 0$ must be a singular point.

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